$X = \mathbb{R}^d$ (not $\mathbb{R}^d \times Y$). Let $\underline{N}_\ell$ be the set of measures $\mu \in \underline{N}$ that are locally finite, i.e. satisfy $\mu(B_n) < \infty$ $\forall n$. Then $\underline{N}_\ell \in \mathcal{N}$.

The palm dist. of a sta. loc. finite pt. pr. η is the dist. of "η conditional on having a pt. at 0" or "η seen from a typical pt. of η". It is defined via the following Refined Campbell Thm

Thm 7.5 Suppose η is a sta. pt pr with intens $\gamma \lambda_d$ $(0<\gamma<\infty)$ (see Prop 7.2). Then $\exists$ unique prob meas P_η^0 on $\underline{N}$ s.t. if η^0 is a pt pr. with dist P_η^0:

$$E \int f(u, T_{-u}\eta)\, \eta(du) = \gamma \int E f(u, \eta^0)\, du \qquad (*)$$

$\forall f \in \mathbb{R}_+(\mathbb{R}^d \times \underline{N}_\ell)$.

or equiv. putting $f(u,\mu) = g(u, T_u \mu)$:

$$E \int g(u, \eta)\, \eta(du) = \gamma \int E\, g(u, T_u \eta^0)\, du \qquad (\overset{*}{*})$$

Remarks ① P_η^0 is the Palm dist. of η

② $(\ast)$ a sort of analogue to Mecke for sta pt pr;

③ Idea of $(\ast)$ for η simple:

$$LHS = E \sum_{du} f(u, T_{-u}\eta) \mathbb{1}_{\eta(du)=1}$$

$$= \sum_{du} \underbrace{P[\eta(du)=1]}_{\downarrow} \; E[f(u, T_{-u}\eta) \mid \eta(du)=1]$$

$$= \int \gamma \, du \quad E[f(u, \eta^0)] \quad RHS$$

ie $\mathcal{L}(T_{-u}\eta \mid \eta(du)=1) = \mathcal{L}(\eta^0)$

④ Put $f(u,\mu) = \mathbb{1}_{\{u \in [0,1]^d\}} \mathbb{1}_{\mu(\{0\})>0}$

LHS of $(\ast)$ is $E\,\eta([0,1]^d) = \gamma$

RHS is $\gamma \int_{[0,1]^d} P[\eta^0(\{0\})>0] \, du$

so $P[\eta^0(\{0\})>0] = 1$.

⑤ Example: put $g(x,\mu) = h(|x - \tau(x,\mu)|)$ $h \in \mathcal{B}_+([0,\infty))$

where for $\mu = \sum_i \delta_{x_i}$ we put $\tau_{x_i}\mu$ = the x_j closest to x_i with $j \neq i$. For $A \subset \mathbb{R}^d$ bounded, $(\ast)$ gives

$$E \int_A h(|x - \tau(x,\eta)|) \, \eta(dx) = \gamma \int_A E\, h(|u - \tau(u, T_u\eta^0)|) \, du$$

$$= \gamma \lambda_d(A) \, E\, h(|\tau(0, T_u\eta^0)|)$$

(6) Suppose η is simple. Then so is η^0

Since putting $f(u,\mu) = 1_{[0,1]^d}(u)\, 1_{\underline{N}_s}(\mu)$

RHS of $(*)$ is $\gamma\, P[\eta^0 \in \underline{N}_s]$

and since η simple $\Rightarrow T_u$ simple $\forall u$

LHS of $(*)$ is $E \int 1_{[0,1]^d}(u) \quad \eta(du)$

$$= \eta[0,1]^d = \gamma$$

So $P[\eta^0 \in \underline{N}_s] = 1$.

<u>Proof of Thm 7.5</u> Given $A \in \mathcal{N}$, $R \in \mathcal{B}(\mathbb{R}^d)$ set

$$f_{A,R}(u,\mu) = 1_R(u)\, 1_A(\mu) \qquad \left(\begin{matrix}\text{show * for}\\ \text{such } f\end{matrix}\right)$$

Fix A:

$R \mapsto E \int f_{A,R}(u, T_{-u}\eta)\,\eta(du)$ is a measure on $\mathbb{R}^d$
call it ν_A. We show ν_A is trans.-invariant
since for $v \in \mathbb{R}^d$, writing $v+R$ for $T_v(R)$:

$$\nu_A(v+R) = E\int 1_{v+R}(u)\, 1_A(T_{-u}\eta)\,\eta(du)$$

$$\stackrel{\text{sta}}{=} E\int 1_R(u-v)\, 1_A(T_{-u+v}\eta)\, T_v\eta(du)$$

$$\stackrel{\$}{=} E\int 1_R(w)\, 1_A(T_{-w}\eta)\,\eta(dw)$$

$$= \nu(R)$$

\$ is by change of variables $w = u - v$, or

formally because for any h:

$$\text{✡} \quad \int h(u-v)\, T_v\eta(dy) = \int h(w)\,\eta(dw)$$

eg $h(u) = 1_B(u)$ so $h(u-v) = 1_{v+B}(u)$ so in ✡

$$\text{LHS} = T_v\eta(v+B) = \eta(B) = \text{RHS}$$

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So $\exists\ \gamma_A : \ \nu_A(R) = \gamma_A \lambda_d(R)$ $\quad R \in \mathcal{B}(\mathbb{R}^d)$

Given R,

$A \mapsto \nu_A(R)$ is a measure on $\mathcal{N}$

Since for A_i disjoint, $f_{\sum_i A_i, R} = \sum_i f_{A_i, R}$

hence $\nu_{\sum A_i}(R) = \sum_i \nu_{A_i}(R)$

So $\gamma_A = \nu_A([0,1]^d)$ is a measure in A.

put $A = \underline{N}$:

$$\gamma_{\underline{N}} = \nu_{\underline{N}}([0,1]^d) = E \int 1_{[0,1]^d}(u)\ \eta(du)$$
$$= \gamma$$

put $P[\eta^0 \in A] = \dfrac{\gamma_A}{\gamma}$. Then

$$E \int f_{A,R}(u, T_{-u}\eta)\ \eta(du) = \nu_A(R)$$

$$= \gamma_A \lambda_d(R) = \gamma\ P[\eta^0 \in A]\ \lambda_d(R)$$

$$= \gamma \int E f_{A,R}(u, \eta^0)\ du$$

So $*$ holds for f of form $f_{A,R}$

hence for all f by a monotone class argument

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uniqueness: if $*$ holds for all f so

(putting $f = f_{A,R}$):

$$E \int_R I_A(T_{-u}\eta)\,\eta(du) = \gamma\,\lambda_d(R)\,P[\eta^0 \in A]$$

so $P[\eta^0 \in A]$ is determined by dist. of η

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Thm 7.6 Suppose η is a sta pt. pr with intensity $0 < \gamma < \infty$. Then η is a PPP iff

$$\eta^0 \overset{d}{=} \delta_0 + \eta.$$

Proof By last part of previous proof: if η is a PPP

$$P[\eta^0 \in A] = \frac{1}{\gamma \lambda_d(R)} E \int_R I_A(T_{-u}\eta) \, d\eta(du)$$

$$\underset{(\text{Thm } 4.1)}{\overset{\text{Mecke}}{=}} \frac{1}{\gamma \lambda_d(R)} \int_R P[T_{-u}(\eta + \delta_u) \in A] \gamma \, du$$

$$= \frac{1}{\lambda_d(R)} \int_R \underbrace{P[T_{-u}\eta + \delta_0 \in A]}_{= P[\eta + \delta_0 \in A]} du$$

$$= P[\eta + \delta_0 \in A] \quad \text{so } \eta^0 \overset{d}{=} \delta_0 + \eta$$

If $\eta^0 \overset{d}{=} \delta_0 + a$ then by (*)

$$E \int g(u,\eta) \, \eta(du) = \gamma \int E g(u, T_u(\overbrace{\delta_0 + \eta}^{\eta^0})) \, du$$

$$= \int E g(u, \delta_u + T_u\eta) \, \gamma \, du$$

$$\overset{\text{sta.}}{=} \int E g(u, \delta_u + \eta) \, \gamma \, du$$

So by Thm 4.3 (Mecke converse), η is a PPP with intensity γdu

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$\underline{N}_s$: {loc & simple measures in $\underline{N}$ }

For $\mu \in \underline{N}_s$ write $x \in \mu$ if $\mu(\{x\}) = 1$.

For $x \in \mu$, $\mu \in \underline{N}_s$ set $\tau(x, \mu)$ = the $y \in \mu$ closest to x (use lex if tie).

Assume now η a sta. pt pr with $\eta \in \underline{N}_s$ a.s.

set $\gamma = E\,\eta([0,1]^d)$, assume $0 < \gamma < \infty$

Suppose $r_n > 0$, $r_n \to 0$ and $B_n = \{x \in \mathbb{R}^d : |x| < r_n\}$.
(here $|\,|$ an arb. norm).

Prop 7.7 $\displaystyle \lim_{n\to\infty} \frac{P[\eta(B_n) \geq 1]}{\lambda_d(B_n)} = \lim_{n\to\infty} \frac{P[\eta(B_n) = 1]}{\lambda_d(B_n)} = \gamma$

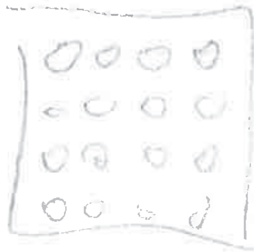
(justifies earlier remark $P(\eta(dx) = 1) = \gamma\, dx$

Proof By Prop. 7.2,

$$(*) \quad \gamma = \frac{E\,\eta(B_n)}{\lambda_d(B_n)} = \underbrace{\frac{P[\eta(B_n) = 1]}{\lambda_d(B_n)}}_{①} + \underbrace{\frac{E[\eta(B_n) \mathbb{1}_{\{\eta(B_n) \geq 2\}}]}{\lambda_d(B_n)}}_{②}$$

We show ② $\to 0$ as $n \to \infty$: Let $B_{n,1}, \dots, B_{n,k_n}$ be disjoint balls of radius r_n contained in $[0,1]^d$, with k_n as large as possible. 68

Then $k_n = \Theta(r_n^{-d})$ so $\exists\, \delta > 0$: (!)

$k_n \geq \delta\, r_n^{-d} \quad \forall n$. Then 

$k_n E[\eta(B_n)\, 1_{\{\eta(B_n) \geq 2\}}]$

$= E \sum_{i=1}^{k_n} \eta(B_{n,i})\, 1_{\{\eta(B_{n,i}) \geq 2\}}$ (by sta.)

$\leq E \int_{[0,1]^d} 1_{\{\eta(B_{2r_n}(x)) \geq 2\}}\, \eta(dx)$

$\xrightarrow{\text{Dom}} E \int_{[0,1]^d} 1_{\eta(\{x\}) \geq 2}\, \eta(dx)$

$= 0$ (η simple)

So ② $\leq$ const. $r_n^{-d}\, E[\eta(B_n)\, 1_{\{\eta(B_n) \geq 2\}}]$

$\leq \quad \| \quad k_n$

$\to 0$

So ① $\to \gamma$

also $\frac{P[\eta(B_n) \geq 2]}{\lambda_d(B_n)} \leq$ ② $\to 0$