Suppose X is a CSMS. A measure λ on X is said to be diffuse if

$$\lambda(\{x\}) = 0 \quad \forall x \in X.$$

A loc. finite pt pr. η on X is called simple if $\eta \in \underline{N}_{sX}$ a.s.

Prop. 6.4 Suppose η is a PPP on X with loc. finite mean measure λ. Then

η is simple $\iff$ λ is diffuse.

Pf $\Rightarrow$ If $\lambda(\{x\}) > 0$, for some x. Then

$$P[\eta \notin \underline{N}_s] \geq P[\eta(\{x\}) > 1] > 0.$$

$\Leftarrow$ 1st suppose $\lambda(X) < \infty$.

Assume $\eta = \sum_{i=1}^{K} \delta_{\xi_i}$

$K \sim Po(\lambda(X))$, ξ_i indep, $P(\xi_i \in \cdot) = \frac{\lambda(\cdot)}{\lambda(X)} = Q(\cdot)$

$$P[\xi_1 = \xi_2] = \iint 1_{\{x=y\}} \, Q(dx) \, Q(dy)$$ $\boxed{5}$

$$= \int Q(\{y\}) \quad Q(dy)$$

$$= 0 \quad \text{since } Q \text{ diffuse.}$$

Hence a.s. $\xi_1, \ldots \xi_K$ are distinct

i.e. η is simple.

In general, take $A_1, A_2, \ldots \in \mathcal{X}$, partitioning X

with $\lambda(A_i) < \infty$. By the above

$\eta|_{A_i}$ is simple for each i.

So η is simple.

Thm 6.5 Renyi's Thm

Suppose η is a simple, locally finite pt pr. On LSMS X, and λ a diffuse measure on X. Suppose

$$P[\eta(B)=0] = \exp(-\lambda(B)) \ \forall B \in \mathcal{X}.$$

Then η is a PPP with mean measure λ.

Remark If we drop the word "simple" can get counterexample by superimposing extra pts on a PPP.

Remark Instead, if we drop "diffuse" the "simplified" PPP(λ) is a counterexample

Earlier ex: $\exists$ pt pr with $\eta(B) \sim Po(\lambda(B))$ $\forall B$ but not PPP. But this λ has atoms (and this η is not simple)

Proof of Th 6.5 Let η' be a PPP with mean meas. λ.

Then $P[\eta(B)=0] = P[\eta'(B)=0] \ \forall B \in \mathcal{X}$.

By next result $\eta \stackrel{d}{=} \eta'$.

Thm 6.6 Suppose η, η' are simple, locally finite pt pr's on CSMS X, ~~from~~ with

$$P[\eta(B)=0] = P[\eta'(B)=0] \quad \forall B \in \mathcal{B}.$$

Then $\eta \stackrel{d}{=} \eta'$.

proof By Prop 5.2 it suffices to show $\eta|_B \stackrel{d}{=} \eta'|_B$ $\forall$ bounded B. Hence may assume $\eta(X) < \infty$ a.s., $\eta'(X) < \infty$ a.s.

Also wlog assume $\cancel{*} \subset [\frac{1}{4}, \frac{3}{4})$ using Thm 6.1

Let $\mathcal{N}'$ be the sub-σ-alg. of $\mathcal{N}$ gen by the collection of sets of the form $\{\mu : \mu(A) = 0\}$

This collection is a Π-system since $(\mu(B)=0 \ \& \ \mu(C)=0) \iff \mu(B \cup C)=0$

By assumption $\mathbb{P}_\eta$ agrees with $\mathbb{P}_{\eta'}$ on this collection

So they agree on $\mathcal{N}'$.

For $A \in \mathcal{H}$, $\mu \in \underline{N}$, $n \in \mathbb{N}$,

$I_{n,j} = ((j-1)/n,\ j/n]$ $\quad 1 \le j \le n$

$$g_{n,A}(\mu) = \sum_{j=1}^{n} \mu(A \cap I_{n,j}) \wedge 1$$

$$g_A(\mu) = \liminf g_{n,A}(\mu)$$

Since η is simple and finite

$\eta(A) = g_A(\eta)$ a.s.

g_A is a $\mathcal{N}'$-meas. fn. on $\underline{N}$

$$P[\eta(A_1)=k_1, \dots, \eta(A_m)=k_m]$$

$$= P[\eta \in \underbrace{\bigcap_{i=1}^{m} g_{A_i}^{-1}(\{k_i\})}_{\in \mathcal{N}'}]$$

$$= P[\eta'(A_1)=k_1, \dots, \eta'(A_m)=k_m] \quad \blacksquare$$

By Th 3.1 $\eta \overset{d}{=} \eta'$.

7 Stationary marked pt processes

Let $d \geq 1$. Let $X = \mathbb{R}^d \times Y$, $(Y, \mathcal{Y})$ a meas space.
We consider pt processes in X (or if Y has 1 element, in $\mathbb{R}^d$)
For $u \in \mathbb{R}^d$ define $T_u : X \to X$

$$T_u(x, y) = (u + x, y)$$

and for $\mu \in \underline{N}$ set $T_u \mu := \mu \circ T_{-u}$ (cf sec 5)

eg $\quad T_u \delta_{(0,y)}(A) = \delta_{(0,y)} \circ T_{-u}(A) = 1\{(0,y) \in T_{-u}(A)\}$

$$= 1\{(u,y) \in A\}$$

$$T_u \delta_{(0,y)} = \delta_{(u,y)}$$

$(T_u : u \in \mathbb{R}^d)$ is a group action on X

— — — — $\underline{N}$

ie $T_u \circ T_v = T_{u+v} \quad \forall u, v$ (flow property)

Def 7.1 A pt process η on X is stationary if

$$T_x \eta \overset{d}{=} \eta \quad \forall x \in \mathbb{R}^d$$

Let λ_d denote Leb. meas on $\mathbb{R}^d$.
The mean meas. of a sta. pt pr is a product of Leb meas λ_d and a meas. on Y:

Prop 7.2 Suppose η is a sta Pt Pr. on $\mathbb{R}^d \times Y$ with

$$E[\eta [0,1]^d \times Y] < \infty$$

Then $\exists$ a finite measure μ on Y such that the mean measure of η is $\lambda_d \otimes \mu$

product meas.

Borel sets

Proof For $A \in \mathcal{Y}$, $B \in \mathcal{B}(\mathbb{R}^d)$ bounded, $u \in \mathbb{R}^d$, the mean measure λ of η satisfies

$$\begin{aligned}\lambda(T_u(B \times A)) &= E\,\eta(T_u(B \times A)) \\ &= E\,\eta(B \times A) \quad (\eta \text{ sta.}) \\ &= \lambda(B \times A)\end{aligned}$$

So $\lambda(\cdot \times A)$ is a trans. Invariant meas on $\mathbb{R}^d$
By well-known fact in Meas Th. $\exists c_A \geq 0$:
$\lambda(\cdot \times A) = c_A \times \lambda_d$

Also for A_i disjoint in $\mathcal{Y}$,

$$C_{\bigcup_{i=1}^{\infty} A_i} = \lambda\left([0,1]^d \times \bigcup_{i=1}^{\infty} A_i\right) = E\,\eta\left([0,1]^d \times \bigcup_{i=1}^{\infty} A_i\right)$$

$$\overset{\text{MON}}{=} \sum_{i=1}^{\infty} E\,\eta\left([0,1]^d \times A_i\right)$$

$$= \sum_{i=1}^{\infty} C_{A_i}$$

So $(C_A, A \in \mathcal{Y})$ is a measure on Y.

Now consider PPP: we get a converse

Prop 7.3 Suppose η is a PPP on $\mathbb{R}^d \times Y$ with intensity λ. If $\lambda = \lambda_d \otimes \mu$ for some finite meas μ on Y, then η is sta.

Proof Let $u \in \mathbb{R}^d$. By the Mapping Theorem (Thm 5.1),

$T_u \eta$ is a PPP in $\mathbb{R}^d \times Y$ with intensity $T_u \lambda$

But for $A \in \mathcal{B}(\mathbb{R}^d)$, $B \in \mathcal{Y}$,

$$T_u \lambda(A \times B) = \lambda(T_u(A \times B))$$

$$= \lambda((-u + A) \times B) \qquad u + A = \{u + x : x \in A\}$$

$$= \lambda_d(-u + A)\,\mu(B) \qquad (\lambda = \lambda_d \otimes \mu)$$

$$= \lambda_d(A)\,\mu(B) = \lambda(A \times B)$$

So $T_x \lambda = \lambda$. ∎

Back to general pt processes:

A non-zero sta pt pr. must be infinite:

Prop 7.4 Suppose η is a sta. pt pr. on $\mathbb{R}^d \times Y$. Then $P[0 < \eta(\mathbb{R}^d \times Y) < \infty] = 0$

Proof Let $E = \{0 < \eta(\mathbb{R}^d \times Y) < \infty\}$. Suppose $P(E) = \delta > 0$.

Let $B_n = \{x \in \mathbb{R}^d : |x| \leq n\}$. Then events

$$E_n = \{\eta(B_n \times Y) = \eta(\mathbb{R}^d \times Y)\} \cap E$$

satisfy $E_n \subset E_{n+1}$ and $\bigcup_{n=1}^{\infty} E_n = E$

so can choose n_0 with $P[E_{n_0}] > \delta/2$

choose $x \in \mathbb{R}^d$ with $|x| > 2n_0$. Let

$$\hat{E}_{n_0} = \{\eta(T_x(B_{n_0} \times Y)) = \eta(\mathbb{R}^d \times Y)\} \cap E.$$

By stationarity $P[\hat{E}_{n_0}] = P[E_{n_0}] > \delta/2$

But $E_{n_0}, \hat{E}_{n_0}$ are disjoint and $E_{n_0} \cup \hat{E}_{n_0} \subset E$, so $P(E) > \delta$ ※