Thm 5.4 Suppose η is a PPP on $\mathbb{X}$ with [4.1] σ-finite mean measure λ, and $C_1, C_2, \ldots$ are disjoint in $\mathcal{X}$. Then

$\eta|_{C_1}, \eta|_{C_2}, \ldots$ are indep PPPs (*)

with mean meas $\lambda|_{C_1}, \lambda|_{C_2}, \ldots$

where $\mu|_C(A) = \mu(A \cap C)$, $A \in \mathcal{X}$.

Proof If $\eta \overset{d}{=} \eta'$ and (*) holds for η' then it also holds for η. By superposition (Thm 4.1) we may take $\eta' = \sum_{i=1}^{\infty} \eta_i'$ where η_i' is a PPP with intensity $\lambda|_{C_i}$ and $(\eta_i', i \geq 1)$ are indep. Then by construction (*) holds for η', and hence also for η.

Thm 5.5 (Colouring of PPP) Suppose

For $i \in \mathbb{N}$, $p_i : X \to [0,1]$ is measurable with $\sum_{i=1}^{\infty} p_i(x) = 1$ $\forall x \in X$.

Suppose η is a PPP on X with σ-finite mean measure λ. Suppose K is a stochastic kernel from X to $\mathbb{N}$ given by

$$K(x, \{j\}) = p_j(x)$$

Then if Θ is the K-marking of η and $\eta_i(\cdot) = \Theta(\cdot \times \{i\})$

then $\eta_1, \eta_2, \ldots$ are indep PPs on X, η_i with mean measure $p_i(x)\lambda(dx)$

Proof By Thm 5.3, Θ is a PPP on $X \times \mathbb{N}$ mean measure $\lambda \otimes K : \lambda \otimes K$.

By Thm 5.4, $\Theta|_{X \times \{1\}}$, $\Theta|_{X \times \{2\}}$, ...

are indep Poisson processes,

and $\Theta|_{X\times\{i\}}$ has mean measure $\lambda\otimes K|_{X\times\{i\}}$. By def., $\forall\, A\in\mathcal{X}$, $i\in\mathbb{N}$,

$$\Theta|_{X\times\{i\}}(A\times\{i\}) = \eta_i(A)$$

and

$$\lambda\otimes\alpha|_{X\times\{i\}}(A\times\{i\}) = \int_A p_i(x)\lambda(dx)$$

So $\eta_1, \eta_2, \dots$ are indep PPPs and η_i has mean measure $p_i(x)\lambda(dx)$,

5 Borel spaces A Borel space is a measurable space $(X, \mathcal{X})$ such that $\exists$ Borel $A \subseteq [0,1]$ and bijection $\phi: X \to A$ with ϕ, ϕ^{-1} measurable (using Borel σ-alg on $[0,1]$)

Thm 5.1 If X is a CSMS (with Borel σ-alg) it is a Borel space

Pf Wlog $0 < d(x,y) < 1$ $\forall x, y \in X$: If not replace by $d(x,y) \wedge 1$, also a metric

Let $\{x_1, x_2, \dots\}$ be dense in X. Let $\alpha: X \to [0,1]^\infty$ be given by $\alpha(x) = (d(x,x_1), d(x,x_2), \dots)$.

Then α is cts: if $y_n \to y$ then $(y_n) \to (y)$

Also $\alpha^{-1}: \alpha(X) \to X$ is cts: if $\alpha(y_n) \to \alpha(y)$ then $\forall \varepsilon > 0$ $\exists$ x_m with $d(y, x_m) < \varepsilon$ so for n large $d(y_n, x_m) < \varepsilon$ so $d(y_n, y) < \varepsilon$

(i) Will show $\alpha(X)$ is measurable in $[0,1]^\infty$

(ii) " " $[0,1]^\infty$ is a Borel space.

(i) Set $U_n = \{ z \in \alpha(X) : z$ has a nbd. $N_{z,n}$ with $\mathrm{diam}(N_{z,n}) < \frac{1}{n}$, $\mathrm{diam}\, \alpha^{-1}(N_{z,n} \cap \alpha(S)) < \frac{1}{n} \}$

$\alpha(X) \subset U_n$ by cty. of α^{-1}

If $x \in U_n$ then $y \in \overline{\alpha(X)}$ close to x is also in $N_{x,n}$

so $y \in U_n$, so U_n is open in $\overline{\alpha(X)}$

Will show $\bigcap_n U_n = \alpha(X)$.

Suppose $x \in \bigcap_n U_n$. Then $x \in \overline{\alpha(X)}$, pick $y_n \in X$

with $\alpha(y_n) \in \bigcap_{k \leq n} N_{x,n}$. Then for $n \leq m$,

$\alpha(y_n) \in N_{x,n}$, $\alpha(y_m) \in N_{x,n}$

$d(y_n, y_m) \leq 1/n$ so $y_n \to y_0$ converg. (X a CSMS)

$d(\alpha(y_0), x) \leq d(\alpha(y_0), \alpha(y_n)) + d(\alpha(y_n), x)$

$\to 0$ since $y_n \to y_0$, $\alpha(y_n) \in N_{x,n}$

So $\alpha(y_0) = x$, $x \in \alpha(X)$, so $\bigcap_n U_n \subset \alpha(X)$

But also $\alpha(X) \subset \bigcap_n U_n$ so $\alpha(X) = \bigcap_n U_n$

U_n' in $\overline{\alpha(S)}$

U_n' open in $[0,1]^\omega$

$\alpha_n(X)$ is Borel in $[0,1]^\omega$.

(ii) $[0,1]^\omega$ is Borel. Consider

$\psi : [0,1] \to [0,1]^\omega$ as follows: For

$x \in I = 0 \cdot x_1 x_2 \cdots$ binary expansion terminates if $\exists$ choice

$\psi(x) = (0 \cdot x_1 x_3 x_5 \cdots, 0 \cdot x_2 x_6 x_{10} \cdots, 0 \cdot x_4 x_{12} x_{20} \cdots, \cdots)$

This ϕ is continuous.
ϕ^{-1} also continuous.

Let β be restr. of ϕ^{-1} to $\alpha(X)$

$\beta \circ \alpha$ is our ϕ_1

Suppose X is a CSMS (and $\mathcal{X}$ the Borel σ-algebra)

A measure λ on X is said to be Diffuse if $\lambda(\{x\}) = 0 \quad \forall \; x \in X$.

Thm 6.2 Suppose η is a locally finite Pt pr. in a (4)

CSMS X (with Borel σ-alg. $\mathcal{X}$). Then

$\exists$ RVs $K, \xi_1, \xi_2, \dots$ K $\mathbb{N}_0 \cup \{\infty\}$-valued

ξ_i X-valued, s.t.

$$\eta = \sum_{i=1}^{K} \xi_i$$

proof 1st assume $\eta(X) < \infty$ a.s. Let ϕ be a measurable bijection

from X to a meas. subset of $[0,1]$

with meas. inverse. Set $\theta = \phi(\eta) = \eta \circ \phi^{-1}$

Then set $K = \eta(X)$ and for $1 \le i \le K$

$$Y_i = \inf\{x : \theta([0,x]) \ge i\} \quad i \le K$$
$$Y_i = 1 \quad i > K$$

then $\theta(A) = \sum_{i=1}^{K} \delta_{Y_i}(A)$ for

all A of the form $(-\infty, x]$, (a π-system)

and hence for all Borel $A \subset [0,1]$

Also for $x<1$: $\{Y_i \le x\} = \{\theta(0,x] \ge i\}$ so Y_i is a RV

Then $\theta = \sum_{1}^{K} \delta_{Y_i}$ and $\eta = \sum_{1}^{K} \delta_{\phi^{-1}(Y_i)}$

doesn't assumption $\eta(X) = \infty$.

Let $x_0 \in X$, $B_n = B(x_0, n)$, $A_n = B_n \setminus B_{n-1}$
($A_1 = B_1$).

Assuming η locally finite

i.e. $\eta(B) < \infty$ $\forall$ bdd B. So can write

$\eta = \sum_{i=1}^{\infty} \eta_i$ with $\eta_i = \eta|_{A_i}$

$\eta_i(A_i) < \infty$ $\forall i$, so by earlier case

$$\eta_i = \sum_{j=1}^{K_i} \delta_{\xi_{ij}}$$

Using $(*)$ & re-numbering, can deduce

$$\eta = \sum_{i=1}^{K} \delta_{x_i}$$ as required.

Suppose X is a CSMS,
A measure λ on X is said to be diffuse (non-atomic) if $\lambda(\{x\}) = 0 \ \forall x \in X$.

A measure $\mu \in \underline{N}$ is called <u>simple</u> if $\mu(\{x\}) = 1 \ \forall x \in X$, and locally finite if $\mu(B) < \infty \ \forall$ bounded $B \in \mathcal{X}$. Let $\underline{N}_s$ be the set of locally finite & simple measures in $\underline{N}$

<u>Prop. 6-3</u> $\underline{N}_s \in \mathcal{N}$.

Let $B_n = B(x_0, n)$ as before.

$$\underline{N}_s = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \underbrace{\{ \mu \in \mathcal{N} : \mu(B_n) = k, \ \mu|_{B_n} \text{ is simple} \}}_{\underline{N}_s^{n,k}}$$

STP $\underline{N}_s^{n,k} \in \mathcal{N}$.

Let ϕ be a measurable bijection from X to a meas. subset of $[0,1)$ with meas. inverse.

(50)

$$\underline{N}_s^{n,k} = \bigcup_{\substack{0<q_1<\cdots<q_{k-1}<1 \\ q_i:\ \text{rational}}} \{\mu \in \underline{N} : \phi(\mu|_{B_n})[0,q_i) = i \text{ for } 1 \le i \le k-1, \text{ and } \mu(B_n) = k\}$$

$$= \bigcup_{\cdots} \{\mu \in \underline{N} : \mu(B_n \cap \phi^{-1}(0,q_i)) = i,\ 1 \le i \le k, \text{ and } \mu(B_n) = k\}$$

$$\in \underline{N}$$