Thm 4.3 Converse to Univariate Mecke:

If λ is a σ-finite measure on $(\mathbb{X}, \mathcal{X})$ and η is a pt. process on $\mathbb{X}$ satisfying

$$E \int_{\mathbb{X}} f(x, \eta) \, \eta(dx) = \int_{\mathbb{X}} E f(x, \eta + \delta_x) \, \lambda(dx) \quad *$$

$\forall f \in \mathbb{R}_+(\mathbb{X} \times \mathbf{N})$, then η is a PPP with mean measure λ.

Ex Given $k \geq 1$, prove or disprove that $\exists$ a converse to k-variate Mecke.

Proof Let $A_1, \cdots A_m$ be disjoint sets in $\mathcal{X}$ with $\lambda(A_i) < \infty$. Let $k_1 \cdots k_m \in \mathbb{N}_0$. Let

$$f(x, \eta) = \mathbb{1}\{x \in A_1, \eta(A_1) = k_1, \cdots, \eta(A_m) = k_m\}$$

Then $E \int_{\mathbb{X}} f(x, \eta) \, \eta(dx) = E[\eta(A_1) \mathbb{1}\{\eta(A_1) = k_1 \cdots \eta(A_m) = k_m\}]$

$$= k_1 P[\eta(A_1) = k_1, \cdots \eta(A_m) = k_m]$$

$$E f(x, \eta + \delta_x) = \mathbb{1}_{A_1}(x) P[\eta(A_1) = k_1 - 1, \eta(A_2) = k_2, \cdots \eta(A_m) = k_m]$$

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So by (*),

$k_1 P[\eta(A_1)=k_1, \cdots \eta(A_m)=k_m]$

$= \lambda(A_1) \, P[\eta(A_1)=k_1-1, \eta(A_2)=k_2, \cdots \eta(A_m)=k_m]$

~~so~~ putting $\pi(k) = P[\eta(A_1)=k \mid \bigcap_{i=2}^{m} \{\eta(A_i)=k_i\}]$

$\pi(k) = \lambda(A_1)\pi(k-1)/k$ so

$\pi(n) = \pi(0)\frac{\pi(1)}{\pi(0)} \cdots \frac{\pi(n)}{\pi(n-1)} = \pi(0)\frac{\lambda(A_1)^n}{n!}$

So $\mathcal{L}(\eta(A_1) \mid \bigcap_{i=2}^{m} \eta(A_i) = k_i)$ is $Po(\lambda(A_1))$

regardless of $k_2 \cdots k_m$ so $\eta(A_1)$ is $Po(\lambda(A_1))$, $\eta(A_2) \cdots \eta(A_m)$

By induction on m,

$\eta(A_1) \cdots \eta(A_m)$ indep.

If $\lambda(A) = \infty$ can write $A = \bigcup_{i=1}^{\infty} A_i$

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with A_i disjoint, $\lambda(A_i) < \infty$, $\sum_{i=1}^{\infty} \lambda(A_i) = \lambda(A) = \infty$.

$$P[\eta(A) \le k] \le P[\eta(\bigcup_{i=1}^{n} A_i) \le k]$$
$$= \sum_{j=1}^{k} P_o(\sum_{i=1}^{n} \lambda(A_i); j)$$
$$\to 0 \text{ as } n \to \infty$$

so $\eta(A) = \infty$ a.s.

Prop 4.4 Suppose N is a $\mathbb{N}_0$-valued RV and $\gamma > 0$. Then $\sim P_o(\gamma)$ iff $\forall f \in \mathbb{R}_+(\mathbb{N}_0)$

$$E[N f(N)] = \gamma f(N+1), \qquad (*)$$

Proof Direct: $\Rightarrow$ ex $\Leftarrow$ take $f(x) = \mathbb{1}_{\{x=k\}}$.

Special case of Thms 4.1 & 4.3. Take X = 1-point set $X = \{x\}$. Then $\eta(X)$ is just a RV N & $f(x,\eta) = g(N)$

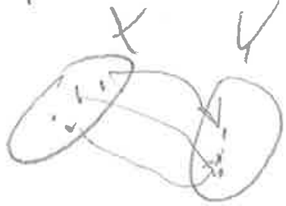
$$\int f(x,\eta)\, \eta(dx) = N g(N)$$
$$\int E f(x, \eta + \delta_x)\, \lambda(dx) = \gamma E g(N+1)$$

Stein chen : if $(*)$ approx holds, N must be approx. Poisson.

5 Mappings, markings & Thinnings

(operations that preserve PPPs)

Suppose $(\mathbb{X}, \mathcal{X})$ & $(\mathbb{Y}, \mathcal{Y})$ are measurable spaces & $T : \mathbb{X} \to \mathbb{Y}$ is measurable. Given pt process



$$\eta = \sum_{n=1}^{\kappa} \delta_{\xi_i} \quad \text{in } \mathbb{X}$$

Let $$T(\eta) = \sum_{n=1}^{\kappa} \delta_{T(\xi_i)} \quad \text{pt process in } \mathbb{Y}$$

$$T(\eta)(A) = \eta(T^{-1}(A)) \qquad A \in \mathcal{Y}$$

i.e. $T(\eta) := \eta \circ T^{-1}$. Use this defn for any pt pr.

<u>Thm 5.1</u> Suppose η is a pt pr. on $\mathbb{X}$ with mean measure λ. Assume the measure $T(\lambda) := \lambda \circ T^{-1}$ is σ-finite. Then

(i) $T(\eta)$ is a pt pr in $\mathbb{Y}$ with mean measure $T(\lambda)$

(ii) If η is a PPP, so is $T(\eta)$.

Proof For $A \in \mathcal{Y}$, $A_i \in \mathcal{Y}$ disjt.

$$T(\eta)(A) = \eta(T^{-1}(A)) \in \mathbb{N}_0 \cup \{\infty\}$$

$$T(\eta)\left(\bigcup_{i=1}^{\infty} A_i\right) = \eta\left(T^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right)\right)$$

$$= \cdots = \sum_{i=1}^{\infty} \eta(\underbrace{T^{-1}(A_i)}_{\text{disj.}})$$

$$= \sum_{i=1}^{\infty} T(\eta)(A_i)$$ so $T(\eta)$ is a pt meas

Also $E[T(\eta)(A)] = E\,\eta(T^{-1}(A)) = \lambda(T^{-1}(A))$

By Assumption $\exists\, A_i$ disjoint with $\bigcup_{i=1}^{\infty} A_i = Y$

and $\lambda(T^{-1}(A_i)) < \infty$

So $\eta(T^{-1}(A_i)) < \infty$ a.s.

$\eta \circ T^{-1}$ is σ-finite a.s.

Finally if $B_1, \dots B_k \in \mathcal{Y}$ are disjoint then

$T^{-1}(B_1) \dots T^{-1}(B_k)$ are disjoint so

$\eta(T^{-1}(B_1)) \dots \eta(T^{-1}(B_k))$ are indep Poisson

Now consider a marked pt processes

given $\eta = \sum_{i=1}^{K} \delta_{\xi_i}$ in X

suppose each pt. ξ_i has

a Y-valued mark attached Y_i

$\mathcal{L}(Y_i \mid \eta)$ depends only on ξ_i, i.e. $P[Y_i \in dy \mid \eta] = K(\xi_i, dy)$

Then $\theta := \sum_{i=1}^{K} \delta_{(\xi_i, Y_i)}$ is a Marked pt pr.

Formally suppose K is a stochastic kernel

i.e. $K : X \times \mathcal{Y} \to [0,1]$ satisfies

(i) $K(x, \cdot)$ is a Prob. measure $\forall x \in X$

(ii) $K(\cdot, A)$ is $\mathcal{X}$-measurable on X, $\forall A \in \mathcal{Y}$

Assume η takes the form $\eta = \sum_{i=1}^{K} \delta_{\xi_i}$ and for

$k \in \mathbb{N}_0$ let $\mathbb{P}_k$ be a choice of the prob. dist of $(\xi_1, \xi_2, \dots \xi_k)$

given $K = k$ so $\mathbb{P}_k$ is a prob dist on X^k

Define $\hat{\mathbb{P}}_k$ on $(X \times Y)^k$ by

$$\tilde{\mathbb{P}}_k(f) = \int f(x_1, \dots x_k, y_1, \dots y_k)$$

$$K(x_1, dy_1) \cdots K(x_k, dy_k)\ \mathbb{P}_k(dx_1, \dots dx_k)$$

(& similar for $k = \infty$). Put

$$\theta = \sum_{i=1}^{K} \delta_{(\xi_i, Y_i)} \quad \text{where}$$

$$\mathbb{P}_{\xi_1, \dots \xi_k, Y_1, \dots Y_k}(dx_1, \dots dx_k, dy_1, \dots dy_k \mid K = k)$$

$$= \tilde{\mathbb{P}}_k(dx_1, \dots dx_k, dy_1, \dots dy_k)$$

Theorem 5.2 Given θ as above,

$$L_\theta(u) = E\, e^{-\theta(u)} = L_\eta(u^*) \qquad u \in B_+(X \times Y)$$

where $u^*(x) = -\log \int e^{-u(x,y)} K(x, dy)$

Hence distribution of θ does not depend on the choice of $\{\xi_i\}$.

Proof $L_\theta(u) = \sum_{k \in [0,\infty]} P[K = k] \int \overbrace{e^{-\sum_{i=1}^{k} u(x_i, y_i)}}^{\prod_{i=1}^{k} e^{-\sum_i u(x_i, y_i)}} K(x_1, dy_1) \cdots K(x_k, dy_k)$

$$\mathbb{P}_k(dx_1, \dots dx_k)$$

$$= \sum_{n \in [0,\infty)} \mathbb{P}[K=n] \int \mathbb{P}_n (dx_1, \dots dx_n) \times \underbrace{\prod_{i=1}^{k} e^{-u^*(x_i)}}_{e^{-\sum u^*(x_i)}}$$

$$= \mathbb{E}\left[e^{-\eta(u^*)} \right] = L_\eta (u^*)$$

Thm 5.3 Suppose η is a PP with σ-finite mean meas λ, then θ (as above) is a PP with mean meas $\lambda \otimes K$:

$$\lambda \otimes K (f) = \int_X \int_Y f(x,y)\, K(x,dy)\, \lambda(dx)$$

$f \in \mathbb{R}_+ (X \times Y)$.

Proof Suppose $\lambda(X) < \infty$. Then may assume

$$\eta = \sum_{i=1}^{K} \delta_{X_i} \qquad K \sim Po(\gamma),\ \gamma = \lambda(X)$$

$$\mathbb{P}[X_i \in \cdot] = \frac{\lambda(\cdot)}{\gamma} =: Q(\cdot)$$

$$\theta = \sum_{i=1}^{K} \delta_{(X_i, Y_i)}$$

where $(X_i, Y_i) \sim Q \times K$ are indep. ⌊ 40

hence by pf of Th 4.1

Θ is a PPP intensity $\gamma\, Q \times K$

$= \lambda \times K$

suppose $\lambda(X) = \infty$. write $\lambda = \sum_{i=1}^{\infty} \lambda_i$ $\lambda_i(X) < \infty$

and $\eta = \sum_{i=1}^{\infty} \eta_i$, η_i PPP with λ_i as mean meas

η_i indep.

putting Θ_i the K-marked pt pr. associated with η_i;

$\Theta = \sum_{i=1}^{\infty} \Theta_i$

PPP with intensity $\sum_{i=1}^{\infty} \lambda_i \otimes K$

$= \lambda \otimes K$.