correction

In Thm (4.1) (superposition) need to assume $\lambda := \sum_{i=1}^{\infty} \lambda_i$ is σ-finite.

5 The Mecke equation

Motivation Consider PP η in $\mathbb{R}^2$ mean meas $\lambda(x)dx$

Let $W \subseteq \mathbb{R}^2$, bounded eg $[0,1]^2$

and writing $\eta = \sum_{i=1}^{\infty} \delta_{\xi_i}$, set

$$Z = \sum_{i : \xi_i \in W} h(D_i) \qquad \text{with } h : \mathbb{R} \to \mathbb{R} \text{ given,}$$

$$D_i := \min |\xi_i - \xi_j| : j \neq i.$$

may Want $\mathbb{E} Z$, $\mathbb{E} Z^2$. Intuition:

$$Z = \sum_{dx \subset W} h(D_x) \mathbb{1}_{\{\eta(dx)=1\}}$$

$$\mathbb{E} Z = \int_W \mathbb{E} h(D_x) \underbrace{\lambda(x)(dx)}_{P[\eta(dx)=1]}$$

D_x = dist x to nearest pt ξ_i

Given $\eta(dx)=1$, the rest of η is distributed as before.

Thm 4.1 (Mecke eq). If λ is a (σ-finite) meas. on $(X, \mathcal{X})$ & η a PPP on X with mean meas. λ, and $f \in \mathbb{R}_+(X \times \underline{N})$ then

$$E \int f(x, \eta)\, \eta(dx) \qquad (*)$$

$$= \int E f(x, \eta + \delta_x)\, \lambda(dx)$$

Remarks (i) if $f(x, \eta)$ depends only on x, this reduces to Campbell's formula for the PPP.

(ii) writing $\eta = \sum_{i=1}^{K} \delta_{\xi_i}$, $\xi_i \in X$, LHS of $(*)$ equals $\sum_{i=1}^{K} f(\xi_i, \eta)$

(like 2 in example)

(iii) Proof: special case of next thm.

In our example,

$$Z^2 = \sum_i \sum_j h(D_i) h(D_j)$$

$$= \underbrace{\sum_i h(D_i)^2}_{Z_1} + \underbrace{\sum\sum_{i \neq j} h(D_i) h(D_j)}_{Z_2}$$

Intuition for $E Z_2$:

$$Z_2 = \sum_{dx} \sum_{\substack{dy \\ y \neq x}} h(D_x) h(D_y) \quad I_{\{\eta(dx)=1, \eta(dy)=1\}}$$

$$E Z_2 = \int_W \int_W E\, h(D_x^y) h(D_y^x)\, \lambda(x) dx\, \lambda(y) dy$$

D_x^y = dist x to nearest pt of $\{\xi_i\} \cup \{y\}$

Defn Given $(X, \mathcal{X})$ & $\mu \in \underline{N}(X)$, define the factorial moment measure $\mu^{(k)} \in \underline{N}(X^k)$ (pt meas. on $X \times X \times \cdots \times X$) as follows.

If $\mu = \sum_i \delta_{x_i}$, set $\mu^{(k)} = \sum_{(i_1, \dots, i_k)}^{\neq} \delta_{(x_{i_1}, \dots, x_{i_k})}$

so for $f \in \mathbb{R}_+(X^k)$

$$\int_{X^k} f\, d\mu^{(k)} = \sum_{i_1} \cdots \sum_{i_k}^{\neq} f(x_{i_1}, \dots, x_{i_k})$$

In general, For $A \subset X^k$ set

$$\mu^{(k)}(A) = \int_X \mu(dx_1) \int_X (\mu - \delta_{x_1})(dx_2)$$

$$\int_X (\mu - dx_1 - dx_2)(dx_3) \cdots \int_X (\mu - \sum_{i=1}^{k-1} \delta x_i)(dx_k)$$

$$\mathbb{1}_A(x_1, \cdots x_k)$$

Thm 4.2 (Multivariate Mecke eqn). Let η be a PPP on X with mean meas λ (σ-finite). Let Let $m \in \mathbb{N}$, $f \in \mathbb{R}_+(X^k \times N)$ [writing $f(x_1, \cdots x_k; \mu)$ for $f((x_1, \cdots x_k), \mu)$]

$$\mathbb{E} \int_{X^k} f(x_1, \cdots x_k; \eta) \quad \eta^{(k)}(d(x_1, \cdots, x_k))$$

$$= \int_X \cdots \int_X E\left[f(x_1, \cdots x_k; \eta + \sum_{i=1}^k \delta_{x_i}) \right] \lambda(dx_1) \lambda(dx_2) \cdots \lambda(dx_k)$$

[special case: $f(x_1, \cdots x_k; \eta) = g(x_1, \cdots x_k)$

$$E \int_{X^k} g(x_1, \cdots x_k) \, \eta^{(k)}(d(x_1, \cdots dx_k))$$

$$= \int_X \cdots \int_X E[g(x_1, \cdots x_k)] \lambda(dx_1) \cdots \lambda(dx_k)]$$

multiple comp ball

<u>Proof</u> Suppose $\lambda(X) = \gamma < \infty$. Let

$(Y_1, Y_2, \cdots)$ be indep X-valued

$P[Y_i \in \cdot] = Q(\cdot) = \frac{\lambda(\cdot)}{\gamma}$, set

$\xi_k = \sum_{i=1}^{k} \delta_{Y_i}$. Then by mixed bin rep

conditioning on value of $\eta(X)$:

$$E \int_{X^k} f(x_1, \cdots, x_k; \eta) \, \eta^{(k)}(d(x_1, \cdots x_k))$$

$$= \sum_{m=k}^{\infty} e^{-\gamma} \frac{\gamma^m}{m!} E\left[\sum^{\neq}_{i_1 \le m, \cdots i_k \le m} f(Y_{i_1}, \cdots Y_{i_k}; \xi_m) \right]$$

indep copy of ξ_{m-k}

$$= \sum_{m=k}^{\infty} e^{-\gamma} \frac{\gamma^m}{m!} \frac{m!}{(m-k)!} E\left[f(Y_1, \cdots Y_k; \sum_{i=1}^{k} \delta_{Y_i} + \xi'_{m-k}) \right]$$

[put $n = m-k$]

$$= \gamma^k \sum_{n=0}^{\infty} e^{-\gamma} \frac{\gamma^n}{n!} E\left[f(Y_1, \cdots Y_m; \sum_{i=1}^{k} \delta_{Y_i} + \xi'_n) \right]$$

$$= \gamma^k \int Q(dy_1) \cdots \int Q(dy_n) \sum_{n=0}^{\infty} \frac{e^{-\gamma} \gamma^n}{n!} E\left[f(y_1, \cdots y_k; \sum_{i=1}^{k} \delta_{y_i} + \xi_n) \right]$$

$$= \int \lambda(dy_1) \cdots \int \lambda(dy_k)\, E\Big[f\big(y_1, \dots, y_k ; \eta + \sum_{i=1}^{k} \delta_{y_i}\big)\Big]$$

now suppose $\lambda(X) = \infty$. Take measures λ_i with $\lambda_i(X) < \infty$, $\lambda = \sum_{i=1}^{\infty} \lambda_i$. Then may assume

$$\eta = \sum_{i=1}^{\infty} \eta_i \qquad \eta_i = \sum_{j=1}^{K_i} \delta_{Y_{ij}}$$

$$Y_{ij} \sim \frac{\lambda_i(\cdot)}{\lambda_i(X)}, \quad K_i \sim Po(\lambda_i(X)).$$

put $\sigma_n = \sum_{i=1}^{n} \eta_i$, $\tau_i = \sum_{i=n+1}^{\infty} \eta_i$.

$$E \int_{X^k} f(x_1, \dots x_k ; \eta)\, \eta^{(k)}(d(x_1, \dots x_k))$$

$$= \lim_{n\to\infty} E \int_{X^k} f(x_1, \dots x_k ; \eta)\, \sigma_n^{(k)} d(x_1 \dots x_k)$$

because $\int \cdots \, d\eta^{(k)} = \sum^{+}_{i_1 j_1 \dots i_k j_k} f(Y_{i_1 j_1} \cdots Y_{i_k j_k} ; \eta)$

with $\sigma_n^{(k)}$, similar but with $i_1 \cdots i_k$ all $\le n$

$$E \int_{X^k} f(x_1, \dots x_k ; \eta)\, \eta^{(k)} (d(x_1 \cdots x_k))$$

$$= \lim_{n \to \infty} E \int_{X^k} f(x_1, \dots x_k ; \sigma_n + \tau_n)\, \sigma_n^{(k)} (d(x_1 \cdots x_k))$$

$$= \lim_{n \to \infty} E \int_{X^k} g_n(x_1, \dots x_k ; \sigma_n)\, \sigma_n^{(k)} d(x_1 \cdots x_k)$$

where $g_n(x_1, \dots x_n, \mu) = E f(x_1, \dots x_k ; \mu + \tau_n)$

$$= \lim_{n \to \infty} \int_{X^k} E g_n(x_1, \dots x_k ; \sigma_n + \sum_{i=1}^{k} \delta_{x_i})\, \tilde{\lambda}_n(dx_1) \cdots \tilde{\lambda}_n(dx_k)$$

where $\tilde{\lambda}_n = \sum_{i=1}^{n} \lambda_i$

$$= \lim_{n \to \infty} \int_{X^k} \underbrace{E\left[f(x_1, \dots x_k ; \eta + \sum_{i=1}^{k} \delta_{x_i}) \right]}_{g(x_1 \cdots x_k)} \tilde{\lambda}_n(dx_1) \cdots \tilde{\lambda}_n(dx_k)$$

$$= \int_{X^k} g(x_1 \cdots x_k)\, \lambda(dx_1) \cdots \lambda(dx_k)$$