Thm 3.1 Given η, η' Pt. processes on $(X, \mathcal{X})$

TFAE:

(i) $\eta(u) \stackrel{d}{=} \eta'(u) \quad \forall\ u \in \mathbb{R}_+(X)$

(ii) $L_\eta(u) = L_{\eta'}(u)$ - - - - $\quad (L_\eta(u) = \mathbb{E}\, e^{-\eta(u)})$

(iii) $\forall\ B_1, \dots B_n \in \mathcal{X}$

$$(\eta(B_1), \dots \eta(B_n)) = (\eta'(B_1), \dots \eta'(B_n))$$

(iv) $\eta \stackrel{d}{=} \eta'$ ie $\mathbb{P}_\eta = \mathbb{P}_{\eta'}$ on $(\underline{N}, \mathcal{N})$

Proof of (iii) $\Rightarrow$ (iv). Let Π be the collection of all subsets of $\underline{N}$ of the form

$$\{\mu \in \underline{N} : \mu(A_1) = k_1, \dots \mu(A_m) = k_m\}$$

with $m \in \mathbb{N}$, $A_1, \dots A_m \in \mathcal{X}$, $k_1, \dots k_m \in \mathbb{N}_0$

Π is a Π-system generating $\mathcal{N}$

if (iii) then $\mathbb{P}_\eta = \mathbb{P}_{\eta'}$ on Π

so $\mathbb{P}_\eta = \mathbb{P}_{\eta'}$ on $\mathcal{N}$, ie (iv).

CSMS: Complete separable metric space

If X is a CSMS, assume $\mathcal{X}$ is the Borel σ-field, we say measure μ on CSMS X is locally finite if $\mu(B) < \infty$ $\forall$ bounded $B \in \mathcal{X}$, and pt. pr. η on X is locally finite if $\eta(B) < \infty$ a.s. $\forall$ bounded $B \in \mathcal{X}$.

Prop 3.2 suppose X is a CSMS, & η, η' are pt processes on X, with $\eta(u) \overset{d}{=} \eta'(u)$ $\forall$ $u \in \mathbb{R}_+(X)$ with bdd support (ie with $\{x : u(x) \neq 0\}$ bdd).

then $\eta \overset{d}{=} \eta'$.

Proof Let $u \in \mathbb{R}_+(X)$. Take $u_n \in \mathbb{R}_+(X)$ bounded support & $u_n \uparrow u$ pointwise

(eg $u_n(x) = u(x)\, \mathbb{1}_{B(x_0, n)}(x)$, some fixed x_0)

By MON, $\eta(u_n) \to \eta(u)$ as $n \to \infty$.

By our assumption and MON,

$$L_\eta(u) = E e^{-u(\eta)}$$
$$= \lim_{n\to\infty} E e^{-u_n(\eta)}$$
$$= \lim_{n\to\infty} E e^{-u_n(\eta')}$$
$$= E e^{-u(\eta')} = L_{\eta'}(u)$$

So $\eta \overset{d}{=} \eta'$

<u>Remark</u> If we assume η, η' are loc. finite but only that $\eta(u) \overset{d}{=} \eta'(u)$ for <u>cts</u> u with bdd support, can show $\eta = \eta'$ (harder).

<u>Remark</u> If X is a csms and η is locally finite, can write

$$\eta = \sum_{n=1}^{K} \delta_{\xi_n}$$

for some sequence of X-valued RVs $\xi_1, \xi_2, \ldots$ and $(\mathbb{N}_0 \cup \{\infty\})$-valued R.V. K.

Proof omitted.

4 The Poisson Process

$(\mathbb{X}, \mathcal{X})$ a measurable space.

Given a σ-finite measure λ on $(\mathbb{X}, \mathcal{X})$ a Poisson process (or Poisson point process or PPP) on $\mathbb{X}$ with intensity measure (mean measure) λ is a pt process η on $\mathbb{X}$ with

(i) $\eta(A) \sim Po(\lambda(A)) \quad \forall A \in \mathcal{X}$

($\eta(A) = \infty$ a.s. if $\lambda(A) = \infty$)

(ii) For $A_1, A_2, \ldots$ disjt.

$\eta(A_1), \ldots \eta(A_n)$ are indep.

By Thm 3.1, if η' also satisfies (i) & (ii) then $\eta \overset{d}{=} \eta'$

[ex (ia) Give example with (i) but not (ii)

(ib) Prove or disprove that we could replace indep by "pairwise indep." in (ii)]

The λ is the intensity measure of η.

Theorem 4.1 (superposition)

Suppose $\lambda_1, \lambda_2, \cdots$ are σ-finite measures on $(X, \mathcal{X})$ and $(\eta_1, \eta_2, \cdots)$ are independent Poisson processes on X with mean measures $\lambda_1, \lambda_2, \cdots$ respectively. Let $\eta(A) = \sum_{i=1}^{\infty} \eta_i(A)$, $A \in \mathcal{X}$. Then η is a PPP with mean measure $\sum_{i=1}^{\infty} \lambda_i(\cdot)$

Proof Let $A_1, \cdots A_k \in \mathcal{X}$ be disjoint. Then $(\eta_i(A_j), 1 \le j \le k, i \in \mathbb{N})$ are a family of indep. RVs, so

$$\left(\sum_i \eta_i(A_1), \sum_i \eta_i(A_2), \cdots, \sum_i \eta_i(A_k) \right)$$

are independent. Also setting $\xi_n = \sum_{i=1}^{n} \eta_i$ we have for $A \in \mathcal{X}$:

(i) $\xi_n(A) = \sum_{i=1}^{n} \eta_i(A) \sim Po\left(\sum_{i=1}^{n} \lambda_i(A)\right)$ 18

(by Prop 1.1)

(ii) $\xi_n(A) \nearrow \eta(A)$ so by continuity of measure, $\forall\ k \in \mathbb{N}$:

$$P[\eta(A) \leq k] = \lim_{n\to\infty} P[\xi_n(A) \leq k]$$

$$= \lim_{n\to\infty} \sum_{j=0}^{k} Po\left(\sum_{i=1}^{n} \lambda_i(A); j\right)$$

$$= \sum_{j=0}^{k} Po\left(\sum_{i=1}^{\infty} \lambda_i(A); j\right)$$

(since $Po(\gamma; j)$ is cts in γ)

so $\eta(A) \sim Po\left(\sum \lambda_i(A)\right)$

<u>Theorem 4.1</u> (existence). Given any σ-finite measure λ on $(\mathcal{X}, \mathscr{X})$ there exists a Poisson process η on $\mathcal{X}$ of intensity λ, and we can write

$$\eta = \sum_{i=1}^{K} \delta_{\xi_i}$$

for some family of $\mathcal{X}$-valued RVs ξ_i and $(\mathbb{N}_0 \cup \{\infty\})$-valued RV K.

<u>proof</u> First suppose $\lambda(\mathcal{X}) < \infty$. On a suitable $(\Omega, \mathcal{F}, P)$ (e.g. a product space) we can arrange to have

(i) a Poisson RV K with parameter $\lambda(\mathcal{X})$

(ii) a sequence $(\xi_1, \xi_2, \xi_3, \cdots)$ of iid $\mathcal{X}$-valued RVs with distribution Q where we put

$$Q(A) = \frac{\lambda(A)}{\lambda(\mathcal{X})}, \quad A \in \mathscr{X}$$

and (ξ_i) are independent of K.

Then set $\eta := \sum_{i=1}^{K} \delta_{\xi_i}$

(call this the mixed binomial representation of a PP).

Claim η is a Poisson process with intensity λ.

Indeed, for $A_1, \dots, A_n \in \mathcal{X}$ partitioning X, setting $p_i = \mathbb{Q}(A_i)$ and $Y_i = \eta(A_i)$:

$K \sim Po(\lambda(X))$ and for $k \in \mathbb{N}_0$

$\mathcal{L}(Y_1, \dots Y_n \mid K = k) = \text{mult}(k; p_1, \dots, p_n)$

So by result from Sec 1, each

$\xi_i \sim Po(\lambda(X) \times p_i) = Po(\lambda(A_i))$

and $\xi_1, \dots \xi_n$ are indep.

Hence η satisfies the defining properties of the PP with intensity λ.

Now suppose $\lambda(\mathbb{X}) = \infty$. Then

choose measures $\lambda_1, \lambda_2, \dots$ on $(\mathbb{X}, \mathcal{X})$

with $\lambda_i(\mathbb{X}) < \infty \ \forall i$ and $\lambda = \sum_{i=1}^{\infty} \lambda_i$

On a suitable (product) prob. space

assume we have indep RVs:

$\kappa_1, \kappa_2, \dots$ and ξ_{ij} $i \in \mathbb{N}, j \in \mathbb{N}$

with $\kappa_i \sim Po(\lambda_i(\mathbb{X}))$ $\mathbb{N}_0$-valued

ξ_{ij} $\mathbb{X}$-valued with dist $\mathbb{Q}_i := \dfrac{\lambda_i(\cdot)}{\lambda_i(\mathbb{X})}$

Put $$\eta = \sum_{i=1}^{\infty} \sum_{j=1}^{\kappa_i} \delta_{\xi_{ij}}$$

$$= \sum_{i=1}^{\infty} \eta_i$$

where $$\eta_i := \sum_{j=1}^{\kappa_i} \delta_{\xi_{ij}}$$

η_i is a PPP with mean measure λ_i

[by $\lambda(\mathbb{X}) < \infty$ case].

$\eta_1, \eta_2, \cdots$ are indep

So by Thm 4.1, η is a PPP with mean measure $\sum_{i=1}^{\infty} \lambda_i$ $\square$

Thm 4.3 (Laplace fnl of PPP)

Let λ be a σ-finite measure on X and η a pt. process on X. Then η is a PPP with mean measure λ iff

$$L_\eta(u) = \exp\left[\int_X (e^{-u(x)} - 1)\lambda(dx)\right]$$

For all $u \in \mathbb{R}_+(X)$

proof "$\Rightarrow$": Suppose η is a PPP with mean measure λ. Then if u is simple, can write $u = \sum_{i=1}^{n} a_i 1_{B_i}$ with $a_i \in (0,\infty)$ and $B_i \in \mathcal{X}$, disjoint. Then

$$L_\eta(u) = E\, e^{-\eta(u)}$$

$$= E\, e^{-\sum a_i \eta(B_i)}$$

$$= \prod_{i=1}^{n} E\, e^{-a_i \eta(B_i)} \qquad \text{[by indep'ce]}$$

$$= \prod_{i=1}^{n} e^{\lambda(B_i)(e^{-a_i}-1)} \qquad \text{[by sec 1]}$$

$$= \exp\left[\sum_{i=1}^{n} (e^{-a_i}-1)\lambda(B_i) \right]$$

$$= \exp\left[\int (e^{-u(x)}-1)\, \lambda(dx) \right]$$

For general $u \in \mathbb{B}_+(X)$, take simple u_n with $u_n \uparrow u$. By MON,

$$L_\eta(u) = \lim_n L_\eta(u_n)$$

$$= \lim_n \exp\left[-\int (1-e^{-u_n(x)})\, d\lambda \right]$$

$$= \exp\left[-\int (1-e^{-u(x)})\, d\lambda \right].$$

~~Mathew Penrose (30-12-11)~~

Duo for violin and cello (violin part)

"$\Leftarrow$" Suppose $\forall u \in \mathbb{R}_+(X)$ that

$$L_\eta(u) = \exp\left[\int (e^{-u(\cdot)} - 1)\, d\lambda\right]$$

Let η' be a PPP with intensity λ.

By "$\Rightarrow$" part,

$$L_{\eta'}(u) = L_\eta(u) \qquad \forall u \in \mathbb{R}_+(X)$$

By Thm 3.1, $\eta \overset{d}{=} \eta'$ so η is also a PPP with intensity λ.