Poisson Processes & Stochastic Geometry

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1 The Poisson distribution

Let $\gamma \geq 0$. A random variable X is Poisson with parameter γ if X takes values in $\mathbb{N}_0 := \{0,1,2,\cdots\}$, and for $k \in \mathbb{N}_0$

$$P[X=k] = \begin{cases} e^{-\gamma}\dfrac{\gamma^k}{k!} := P_0(\gamma;k) & \gamma > 0 \\ \mathbb{1}_0(k) & \gamma = 0 \end{cases}$$

For $s \geq 0$, $E[s^X] = \sum_{k=0}^{\infty} s^k e^{-\gamma}\frac{\gamma^k}{k!} = e^{-\gamma}e^{s\gamma} = e^{\gamma(s-1)}$ (MGF)

So for $t \geq 0$ $E[e^{-tX}] = e^{\gamma(e^{-t}-1)}$ (Laplace Tr)

Also $E[X] = \gamma$, $E[(X)_k] = \gamma^k$ where $(n)_k = n(n-1)\cdots(n-k+1)$

<u>Prop. 1.1</u> Let $\gamma, \delta \geq 0$ & suppose X, Y are indep. $P_0(\gamma)$, $P_0(\delta)$ resp.

Then ① $X+Y \sim P_0(\gamma+\delta)$,

② $\mathcal{L}(X \mid X+Y=k) \sim \mathrm{Bin}\left(k, \frac{\gamma}{\gamma+\delta}\right)$ if $\gamma+\delta > 0$

<u>Proof</u> Set $p = \frac{\gamma}{\gamma+\delta}$. For $j, k \in \mathbb{N}_0$,

$$P[X=j, X+Y=k] = \frac{e^{-\gamma}\gamma^j}{j!}\frac{e^{-\delta}\delta^{k-j}}{(k-j)!}$$

$$= e^{-(\gamma+\delta)}\frac{(\gamma+\delta)^k}{k!}\binom{k}{j}\left(\frac{\gamma}{\gamma+\delta}\right)^j\left(\frac{\delta}{\gamma+\delta}\right)^{k-j}$$

Sum over j to get ①

Then get ②.

Next result is a sort of converse

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<u>Prop 1.2</u> (Poisson thinning). Let $\lambda > 0$, $p \in [0,1]$. If $Z \sim Po(\lambda)$ and

$$\mathcal{L}(X \mid Z = k) = Bin(k, p) \qquad \forall k \in \mathbb{N}_0$$

then $X \sim Po(\lambda p)$ and X is indep. of $Z - X$

<u>Pf</u> $\forall j, k \in \mathbb{N}_0$

$$P[X = j, Z - X = k] = P[Z = j+k] P[X = j \mid Z = j+k]$$

$$= \frac{e^{-\lambda} \lambda^{j+k}}{(j+k)!} \binom{j+k}{j} p^j (1-p)^k$$

$$= P_o(\lambda p ; j) P_o(\lambda(1-p) ; k)$$

<u>Extension</u> if $Z \sim Po(\lambda)$ & for some $p_1, \dots p_m \geq 0$, $\sum p_i = 1$ &

$$\mathcal{L}((X_1, \dots X_m) \mid Z = k) = mult(k ; p_1, \dots, p_m)$$

then $X_1, \dots X_m$ are indep $X_i \sim Po(p_i)$

(ex).

(2) **Point processes.** Let $(X, \mathcal{X})$ be a measurable space (i.e. $\mathcal{X}$ is a σ-field on the set X). (eg $\mathbb{R}^n$)

We aim to model a random set of points in X

Could consider a set of X-valued RVs $\{\xi_1, \xi_2, \dots\}$

Don't care about order; may have multiplicity

We'll represent this set as a point measure

$$\eta(A) = \#\text{ points in } A \qquad A \in \mathcal{X}$$

Given $(X, \mathcal{X})$ Let $\underline{N} = \underline{N}(X)$ be the class of σ-finite measures μ on $(X, \mathcal{X})$ such that $\mu(B) \in \mathbb{N}_0 \cup \{\infty\}$ $\forall B \in \mathcal{X}$.

e.g. Dirac measure δ_x: for $x \in X$

$$\delta_x(B) = \mathbb{1}_B(x) = \begin{cases} 1 & x \in B \\ 0 & \text{if not} \end{cases}$$

For $x_1, \dots x_n$ in $\mathcal{X}$

$$\sum_{i=1}^{n} \delta_{x_i} \in \underline{N}.$$

Let $\mathcal{N}$ be the σ-algebra on $\underline{N}$:

$$\mathcal{N} = \sigma\left(\{\mu \in \underline{N} : \mu(B) = k\}\ B \in \mathcal{X}, k \in \mathbb{N}_0\right)$$

A point process on $\mathbb{X}$ is a measurable function

$$\eta : \Omega \to \mathbf{N}$$

(for some prob. space $(\Omega, \mathcal{F}, \mathbb{P})$)

Often we write η for $\eta(\omega)$

$\eta(B)$ for $\eta(\omega)(B)$

$\eta(B)$ is a R.V. for $B \in \mathcal{X}$.

Example Suppose Q is a Prob. measure on $(X, \mathcal{H})$ and $\xi_1, \xi_2, \dots \xi_n$ are i.i.d. Q-distributed random elements of X on $(\Omega, \mathcal{F}, P)$. Setting

$$\eta = \sum_{i=1}^{n} \delta_{\xi_i}$$

gives a point process η.

$$\eta(A) = \# \text{ pts in } A$$

check η measurable, suffices to consider $n=1$.
Then $\forall A \in \mathcal{H}$

$$\{\omega : \eta(\omega) = 1\} = \{\omega : X_1(\omega) \in A\} \in \mathcal{F} \quad \checkmark$$

Note this η satisfies

$$\eta(A) \sim \mathrm{Bin}(n, Q(A))$$

and is called a binomial <u>point process</u>

Given pt. pr. η on $(\mathbb{X}, \mathcal{X})$, its
<u>intensity</u> is the measure λ on $(\mathbb{X}, \mathcal{X})$:

$$\lambda(A) = \mathbb{E}[\eta(A)]$$

ex. λ is a measure

ex. For Binomial process above,

$$\lambda(A) = n \, \mathbb{Q}(A)$$

A [illegible] measurable [illegible]

[illegible]

<u>Notation</u> $\mathbb{R}(X) = \{ \mathcal{A}\text{-measurable fns } X \to \mathbb{R} \}$

$\bar{\mathbb{R}}(X) = \{ - - - - - - X \to [-\infty, \infty]$

$\mathbb{R}_+(X) = \{ - - - - - X \to [0, \infty)$

$\bar{\mathbb{R}}_+(X) = \{ - - - - - X \to [0, \infty]$

For any measure μ on $(X, \mathcal{A})$ and $u \in \bar{\mathbb{R}}(X)$

define $\int u \, d\mu = \int u^+ d\mu - \int u^- d\mu$

with convention $\infty - \infty = 0$.

<u>Theorem</u> 2.1 (Campbell's formula)

If η is a point process on $(\mathbb{X}, \mathcal{X})$ with intensity λ, and $u \in \bar{\mathbb{R}}(\mathbb{X})$, then $\int_{\mathbb{X}} u \, d\eta$ is a random variable with

$$\mathbb{E}\left[\int_{\mathbb{X}} u \, d\eta\right] = \int_{\mathbb{X}} u \, d\lambda \qquad (*)$$

provided either $u \geq 0$ or $\int_{\mathbb{X}} |u(x)| \, \lambda(dx) < \infty$.

<u>Exercise</u> Give an example where $(*)$ fails (so the condition on the last line must fail too)

<u>Proof of Thm 2.1</u> suppose $u = \mathbb{1}_A$, $A \in \mathcal{X}$.

Then $\int_{\mathbb{X}} u \, d\eta = \eta(A)$, a R.V.

$$E \int_{\mathbb{X}} u \, d\eta = \lambda(A) = \int_A u \, d\lambda$$

Extend to simple u by linearity, then to nonnegative u by MON, then to general u by taking +ve and -ve parts.

(3) Distributions and their characterization

Given random elements ξ, ξ' of a measurable space $(Y, \mathcal{Y})$, we write $\xi \overset{d}{=} \xi'$ if the measures $\mathbb{P}_\xi$ on $(Y, \mathcal{Y})$ given by $\mathbb{P}_\xi(\cdot) = \mathbb{P}[\xi \in \cdot]$, coincides with $\mathbb{P}_{\xi'}$. We call $\mathbb{P}_\xi$ the distribution of ξ. A standard result in measure theory tells us that for $f \in \mathbb{R}_+(Y)$, $\mathbb{E}[f(\xi)] = \int_Y f \, d\mathbb{P}_\xi$.

In particular, the distribution of a point process η on $(\mathbb{X}, \mathcal{X})$ is defined as the measure $\mathbb{P}_\eta$ on $(\underline{N}, \mathcal{N})$ given by $\mathbb{P}_\eta(\cdot) = \mathbb{P}[\eta \in \cdot]$

Given a pt. process η on $(\mathcal{X}, \mathscr{X})$, the Laplace functional of η is the mapping $L_\eta : \overline{\mathbb{R}}_+(\mathcal{X}) \to [0,1]$:

$$L_\eta(u) = \mathbb{E}\, e^{-\eta(u)}$$

where we sometimes write $\lambda(u)$ for $\int_{\mathcal{X}} u \, d\lambda$ for $u \in \overline{\mathbb{R}}(\mathcal{X})$, λ a measure on $(\mathcal{X}, \mathscr{X})$.

Theorem 3.1 For pt. processes η, η' on $(\mathcal{X}, \mathscr{X})$ TFAE:

(i) $\eta(u) \overset{d}{=} \eta'(u)$ as RVs, $\forall u \in \overline{\mathbb{R}}_+(\mathcal{X})$

(ii) $L_\eta(u) = L_{\eta'}(u)$ $\forall u \in \overline{\mathbb{R}}_+(\mathcal{X})$.

(iii) $(\eta(B_1), \dots, \eta(B_n)) \overset{d}{=} (\eta'(B_1), \dots \eta'(B_n))$ $\forall n \in \mathbb{N}, B_1, \dots B_n \in \mathscr{X}$

(iv) $\eta \overset{d}{=} \eta'$

In particular, the Laplace functional characterizes the distribution of a point process.

<u>Proof</u> (iv) $\Rightarrow$ (i). Given $u \in \overline{\mathbb{R}}_+(X)$,

$g_u : \xi \mapsto \int u \, d\xi$ is measurable $\underline{N} \to \mathbb{R}$

$\mathbb{P}_{\eta(u)}(\cdot) = \mathbb{P}[\eta(u) \in \cdot] = \mathbb{P}[g_u(\eta) \in \cdot] = \mathbb{P}[\eta \in g_u^{-1}(\cdot)]$

and $\mathbb{P}_{\eta'(u)}(\cdot) = \mathbb{P}[\eta' \in g_u^{-1}(\cdot)]$ similarly.

So if $\mathbb{P}_\eta = \mathbb{P}_{\eta'}$ then $\mathbb{P}_{\eta(u)} = \mathbb{P}_{\eta'(u)}$

(i) $\Rightarrow$ (ii) For any R.V. Y, $\mathbb{E} e^{-Y} = \int_{\mathbb{R}} e^{-y} \, d\mathbb{P}_Y(y)$

which is determined by the distribution of Y.

So if (i) then $L_\eta(u) = \mathbb{E} e^{-\eta(u)} = \mathbb{E} e^{-\eta'(u)} = L_{\eta'}(u)$

for all u, i.e. (ii).

(ii) $\Rightarrow$ (iii) Assume (ii). Take $u(x) = \sum_{i=1}^{n} a_i \mathbb{1}_{B_i}(x)$

with $a_i > 0$.

$$L_\eta(u) = \mathbb{E} \exp\Big(-\sum a_i \underbrace{\eta(B_i)}_{\xi_i}\Big)$$

$$= \hat{\mathbb{P}}_{\xi_1, \cdots \xi_n}(a_1, \cdots a_n),$$ where $\hat{\mathbb{P}}_{\xi_1, \cdots \xi_n}$

the multivariate laplace transform (MVLT)

the distribution of $\xi_1, \cdots \xi_n$.

[Given measure μ on $[0,\infty]^n$ define MVLT

$$\hat{\mu}(a_1,\cdots a_n) = \int_{[0,\infty]^n} e^{-a \cdot x} \mu(dx) \quad , (a_i > 0)]$$

MVLT determine measures on $[0,\infty)^+$ so by the above $\mathbb{P}_{\xi_1,\cdots\xi_n} = \mathbb{P}_{\xi_1',\cdots\xi_n'}$ where we put $\xi_i' = \eta'(B_i)$. That is, (iii) holds.

solution (counterexample to Campbell's formula)

put $u = \begin{cases} 1 & \text{on } (0,1) \\ -1 & \text{on } (-1,0] \end{cases}$

Let N be a RV with $\mathbb{E}[N] = \infty$ ($N \in \mathbb{N}$). Put

$$\eta = N\delta_{1/2} + (N-1)\delta_{-1/2}$$

$$\int u \, d\eta = N - (N-1) = 1$$

$$\int u \, d\lambda = \infty - \infty = 0$$