49. Let $-\infty < a < b < \infty$. Suppose $g : [a, b] \to \mathbb{R}$ is a continuously differentiable, strictly increasing function. Show that for all bounded Borel-measurable $f : (a, b] \to \mathbb{R}$ we have the change of variables formula $\int_{g(a)}^{g(b)} f(y)dy = \int_a^b f(g(x))g'(x)dx$.

    Hint: First verify this for $f = \mathbf{1}_{(g(a),t]}$ with $g(a) < t \leq g(b)$. Then use the Monotone Class theorem.

    Unfortunately, there is a typo in the question: It should be for $f : (g(a), g(b)] \to \mathbb{R}$ rather than for $f : (a, b] \to \mathbb{R}$.

    Let $\mathcal{H}$ be the class of bounded measurable functions $f : (g(a), g(b)] \to \mathbb{R}$ such that $\int_{g(a)}^{g(b)} f(y)dy = \int_a^b f(g(x))g'(x)dx$.

    First suppose $f = \mathbf{1}_{(g(a),t]}$ with $g(a) < t \leq g(b)$. Then by definition 10.4 and Lemma 10.7(b),

$$\int_{g(a)}^{g(b)} f(y)dy = \int_{\mathbb{R}} \mathbf{1}_{(g(a),g(b)]}\mathbf{1}_{(g(a),t]}d\lambda_1 = \int_{\mathbb{R}} \mathbf{1}_{(g(a),t]}d\lambda_1 = \lambda_1((g(a), t]) = t - g(a).$$

    Also by the Fundamental Theorem of Calculus,

$$\int_a^b f(g(x))g'(x)dx = \int_a^{g^{-1}(t)} g'(x)dx = t - g(a)$$

    and therefore $\mathbf{1}_{(g(a),t]} \in \mathcal{H}$.

    Set $W = (g(a), g(b)]$. The class of sets $\mathcal{D} = \{(g(a), t] : g(a) < t \leq g(b)\}$ is a π-system, and it generates $\mathcal{B}_W$ by Question 36. By our previous argument $\mathbf{1}_A \in \mathcal{H}$ for all $A \in \mathcal{D}$.

    If $f, h \in \mathcal{H}$ and $\alpha \in \mathbb{R}$ then $f, h \in L^1((g(a), g(b)])$ (since they are bounded) and by linearity of integration (Theorem 11.5)

$$\int_{g(a)}^{g(b)} (f+h)(y)dy = \int_{g(a)}^{g(b)} f(y)dy + \int_{g(a)}^{g(b)} h(y)dy$$
$$= \int_a^b f(g(x))g'(x)dx + \int_a^b h(g(x))g'(x)dx = \int_a^b (f+h)(g(x))g'(x)dx$$

    so $f + h \in \mathcal{H}$, and also by linearity

$$\int_{g(a)}^{g(b)} \alpha f(y)dy = \alpha \int_{g(a)}^{g(b)} f(y)dy = \alpha \int_a^b f(g(x))g'(x)dx = \int_a^b \alpha f(g(x))g'(x)dx$$

    so $\alpha f \in \mathcal{H}$.

    If $f_n \in \mathcal{H}$ for $n \in \mathbb{N}$ with $0 \leq f_n \uparrow f$ pointwise, and f is bounded, then by MON,

$$\int_{g(a)}^{g(b)} f(y)dy = \lim_{n\to\infty} \int_{g(a)}^{g(b)} f_n(y)dy = \lim_{n\to\infty} \int_a^b f_n(g(y))g'(y)dy = \int_a^b f(g(y))g'(y)dy$$

    so $f \in \mathcal{H}$. Therefore we can apply the Monotone Class Theorem (Theorem 13.1) to deduce that all bounded $\mathcal{B}_W$-measurable functions on W are in the class $\mathcal{H}$, as required.

50. (a) Show that $\{(x,y) \in \mathbb{R}^2 : x < y\} \in \mathcal{B} \otimes \mathcal{B}$.

(b) Let $c \in (0,\infty)$. Show that $\{(x,y) \in \mathbb{R}^2 : x < y \le x + c\} \in \mathcal{B} \otimes \mathcal{B}$.

(c) Suppose μ is a probability measure on $(\mathbb{R}, \mathcal{B})$. For $x \in \mathbb{R}$, let $F(x) = \mu((-\infty, x])$. Let $c \in \mathbb{R}$. Use Fubini's Theorem to show that $\int_{-\infty}^{\infty}(F(x+c) - F(x))dx = c$.

(a) Given $b \in \mathbb{R}$ let $B_b := \{(x,y) \in \mathbb{R}^2 : x + b < y\}$. Then

$$B_b = \cup_{q \in \mathbb{Q}}\{(x,y) \in \mathbb{R}^2 : x + b < q < y\} = \cup_{q \in \mathbb{Q}}((-\infty, q-b) \times (q, \infty)),$$

so that B_b is a countable union of sets in $\mathcal{B} \otimes \mathcal{B}$ and therefore is itself in $\mathcal{B} \otimes \mathcal{B}$.

Taking $b = 0$ gives us the result.

(b) With B_b as defined above, given $c \in (0, \infty)$ we have

$$\{(x,y) \in \mathbb{R}^2 : x < y \le x + c\} = B_0 \setminus B_c \in \mathcal{B} \otimes \mathcal{B}.$$

(c) First suppose $c \ge 0$. We shall apply Fubini's theorem (in fact Tonelli's theorem) to the product of the measure spaces $(\mathbb{R}, \mathcal{B}, \lambda_1)$ (where λ_1 is Lebesgue measure) and $(\mathbb{R}, \mathcal{B}, \mu)$. For $(x,y) \in \mathbb{R} \times \mathbb{R}$ set $f(x,y) = 1$ if $x < y \le x + c$ (or equivalently, if $y - c \le x < y$), and otherwise $f(x,y) = 0$.

Then $f \ge 0$ and f is $(\mathcal{B} \otimes \mathcal{B})$-measurable by part (b), so we can apply Tonelli's theorem. We have

$$\int_{\mathbb{R}}\left[\int_{\mathbb{R}} f(x,y)\mu(dy)\right]\lambda_1(dx) = \int_{\mathbb{R}}\left[\int_{\mathbb{R}} \mathbf{1}_{(x,x+c]}(y)\mu(dy)\right]\lambda_1(dx)$$
$$= \int_{-\infty}^{\infty}[\mu((-\infty, x+c] - \mu((-\infty, x])]dx = \int_{-\infty}^{\infty}[F(x+c) - F(x)]dx \tag{3}$$

and by Tonelli's theorem, this is equal to the same double integral taken in the opposite order, which comes to

$$\int_{\mathbb{R}}\left[\int_{\mathbb{R}} f(x,y)\lambda_1(dx)\right]\mu(dy) = \int_{\mathbb{R}}\left[\int_{\mathbb{R}} \mathbf{1}_{[y-c,y)}(x)\lambda_1(dx)\right]\mu(dy) = \int_{\mathbb{R}} c\mu(dy) = c \tag{4}$$

where the last equality is because μ is a probability measure. The equality between the expressions (3) and (4) gives us the result, for the case $c > 0$.

Now we consider the case with $c < 0$. Put $C = -c$, and set $G(x) = F(x+c) - F(x)$. Then by Question 37,

$$\int_{-\infty}^{\infty} G(x)dx = \int_{-\infty}^{\infty} G(x+C)dx = \int_{-\infty}^{\infty}(F(x) - F(x+C))dx = -\int_{-\infty}^{\infty}(F(x+C) - F(x))dx$$

and by the case of this result that we already proved, the last expression equals $-C = c$ as required.

51. Let $A \subset \mathbb{R}^2$ be a Borel set, and for $x \in \mathbb{R}$ let $A_x := \{y \in \mathbb{R} : (x,y) \in A\}$. Show that

$$\lambda_2(A) = \int_{-\infty}^{\infty} \lambda_1(A_x)dx,$$

where λ_d denotes d-dimensional Lebesgue measure.

We have $\lambda_2(A) = \int_{\mathbb{R}^2} \mathbf{1}_A d\lambda_2$. Therefore since $\lambda_2 = \lambda_1 \otimes \lambda_1$ and $\mathcal{B}_2 = \mathcal{B}_1 \otimes \mathcal{B}_1$, by Tonelli's theorem

$$\begin{aligned}\lambda_2(A) &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathbf{1}_A(x,y)dy \right) dx \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathbf{1}_{A_x}(y)dy \right) dx \\ &= \int_{\mathbb{R}} \lambda_1(A_x)dx.\end{aligned}$$

52. For $A \subset \mathbb{R}^d$ and $u \in \mathbb{R}^d$ let $A + u := \{a + u : a \in A\}$. Also if $d = 2$, for $x \in \mathbb{R}$ set $A_x := \{y \in \mathbb{R} : (x,y) \in A\}$.

    (a) Let $-\infty < a < b < \infty$, and let $I = (a,b)$. Let $y \in (0,\infty)$. Compute $\lambda_1((I+y) \setminus I)$.

    (b) Let $B \subset [0,1]^2$ and suppose B is open and B is convex, i.e. for all $u, v \in B$ and $\alpha \in (0,1)$ we have $\alpha u + (1-\alpha)v \in B$. Let e be the unit vector $(0,1)$ and for $t > 0$ let $B(t) := B + te$. Given $x \in \mathbb{R}$, show that $B(t)_x = B_x + t$.

    (c) Show that $\lambda_1((B(t) \setminus B)_x) = \min(t, \lambda_1(B_x))$.

    (d) Show that $\lambda_2(B(t) \setminus B) \leq t$.

    (e) Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ denote projection onto the first co-ordinate, i.e. $\pi(x,y) = x$. Show that $t^{-1}\lambda_2(B(t) \setminus B) \to \lambda_1(\pi(B))$ as $t \downarrow 0$.

(a) If $y < b - a$ then $(I + y) \setminus I = [b, b + y)$ so $\lambda_1((I + y) \setminus I) = y$.

If $y \geq b - a$ then $(I + y) \setminus I = (a + y, b + y)$ so $\lambda_1((I + y) \setminus I) = b - a$.

In other words, $\lambda_1((I + y) \setminus I) = \min(y, b - a)$.

(b) We have that

$$\begin{aligned}y \in B(t)_x &\iff (x,y) \in B + (0,t) \iff (x, y - t) \in B \\ &\iff y - t \in B_x \iff y \in B_x + t\end{aligned}$$

so $B(t)_x = B_x + t$ as claimed.

(c) Since B is convex, B_x is also convex and therefore an interval (or the empty set), by Parts (a) and (b) we have that

$$\lambda_1((B(t) \setminus B)_x) = \lambda_1(B(t)_x \setminus B_x) = \lambda_1((B_x + t) \setminus B_x) = \min(t, \lambda_1(B_x)).$$

(d) Let $A(t) := B(t) \setminus B$. For $x \notin [0,1]$, $(A(t))_x = \varnothing$. Also by Question 26, since B is open, $B \in \mathcal{B}_2$, and likewise $B(t) \in \mathcal{B}_2$, so $A(t) \in \mathcal{B}_2$. Therefore by Question 51 and then Part (c),

$$\lambda_2(A(t)) = \int_{-\infty}^{\infty} \lambda_1(A(t)_x)dx = \int_0^1 \min(t, \lambda_1(B_x))dx \leq \int_0^1 tdx = t.$$

(e) As shown in the solution to (d), we have $t^{-1}\lambda_2(A(t)) = \int_0^1 g_t(x)dx$, where we set $g_t(x) = t^{-1}\min(t, \lambda_1(B_x))$.

Let $t_n \downarrow 0$. Then as $n \to \infty$ we have $g_{t_n}(x) \to g(x)$ where $g = \mathbf{1}_{\pi(B)}$. This is because if $x \notin \pi(B)$ then $B_x = \varnothing$ so $g_t(x) = 0$ for all t, but if $x \in \pi(B)$ then B_x is an open non-empty inteval so $\lambda_1(B_x) > 0$ so $g_t(x) = 1$ for t small.

Also $g_{t_n}(x) \leq g_{t_{n+1}}(x)$ for all n, x, so by MON, $\int_0^1 g_{t_n}(x)dx \to \int g(x)dx = \lambda_1(\pi(B))$ as $n \to \infty$. By the hint this gives us the result.

53. Let $(X, \mathcal{M})$ be a measurable space and suppose $f : X \to [0, \infty]$ and $g : X \to [0, \infty]$ are Borel functions. Show that

$$\int_0^\infty \int_0^\infty \mu(\{x \in X : f(x) > s, g(x) > t\})dsdt = \int_X f(x)g(x)\mu(dx).$$

By Question 34, the set $A \subset X \times \mathbb{R} \times \mathbb{R}$ given by $A = \{(x, s, t) : f(x) > s, g(x) > t\}$ is in $\mathcal{M} \otimes \mathcal{B} \otimes \mathcal{B}$. Therefore the function $\mathbf{1}_A$ is measurable with respect to $\mathcal{M} \otimes \mathcal{B} \otimes \mathcal{B}$, by Question 32. Therefore by Tonelli's theorem,

$$\begin{aligned}
\int_0^\infty \int_0^\infty \mu(\{x \in X : f(x) > s, g(x) > t\})dsdt &= \int_0^\infty \int_0^\infty \int_X \mathbf{1}_A((x, s, t))\mu(dx)dsdt \\
&= \int_X \int_0^\infty \int_0^\infty \mathbf{1}_A((x, s, t))dsdt\mu(dx) \\
= \int_X \int_0^\infty \int_0^\infty & \mathbf{1}_{[0,f(x))}(s)\mathbf{1}_{[0,g(x))}(t)dsdt\mu(dx) \\
& \int_X \int_0^\infty f(x)\mathbf{1}_{[0,g(x))}(t)dt\mu(dx) \\
& \int_X f(x)g(x)\mu(dx).
\end{aligned}$$

54. (a) Let $\alpha \in \mathbb{R}$ be a fixed constant. Let $f(x) = x^\alpha$ for $x \in (0, 1]$. Determine the values of $p \in [1, \infty)$ (depending on α), such that $f \in L^p([0, 1])$.
(b) Let $\alpha \in \mathbb{R}$, and let $g(x) = x^\alpha$ for $x \in [1, \infty)$. Determine the values of $p \in [1, \infty)$ (depending on α) such that $g \in L^p([1, \infty))$.

(a) To have $f \in L^p([0, 1])$ we need to have $\int_0^1 |f(x)|^\alpha dx < \infty$. Since $f(x) = x^\alpha > 0$, this condition amounts to $\int_0^1 x^{\alpha p}dx < \infty$. If $\alpha p > -1$ then $\int_0^1 x^{\alpha p}dx = [x^{\alpha p+1}]_0^1/(\alpha p + 1) < \infty$. If $\alpha p < -1$, we get the same indefinite integral but now the integral comes to $+\infty$ because $x^{\alpha p+1}$ diverges at $x = 0$. If $\alpha p = -1$ then $\int_0^1 x^{\alpha p}dx = [\log x]_0^1 = +\infty$. To sum up, $f \in L^p([0, 1])$ if and only if $\alpha p > -1$.

Remark. To go into a bit more detail with the argument above, observe that $\int_0^1 x^{\alpha p}dx = \lim_{n\to\infty} \int_{1/n}^1 x^{\alpha p}dx$ by MON, and the function $x^{\alpha p}$ is bounded and continuous on $[1/n, 1]$ so by Theorem 13.8, the Lebesgue integral $\int_{1/n}^1 x^{\alpha p}dx$ equals the Riemann integral, so by the fundamental theorem of calculus it equals $g(1)-g(1/n)$ where $g(x) = \log x$ if $\alpha p = -1$, or $g(x) = x^{\alpha p+1}/(\alpha p+1)$ for $\alpha p \neq -1$. Thus $\int_0^1 f(x)dx = \lim_{n\to\infty}(g(1) - g(1/n))$, which is finite if $\alpha p > -1$, and infinite if $\alpha p \leq -1$.

(b) Similarly to (a), to have $g \in L^p([1,\infty))$ we need $\int_1^\infty x^{\alpha p}dx < \infty$. If $\alpha p = -1$ then $\int_1^\infty x^{\alpha p}dx = [\log x]_1^\infty = +\infty$. If $\alpha p \neq -1$ then $\int_1^\infty x^{\alpha p}dx = [x^{\alpha p+1}]_1^\infty/(\alpha p+1)$ which is finite if $\alpha p < -1$ but infinite if $\alpha p > -1$. So this time we have $g \in L^p([1,\infty))$ if and only if $\alpha p < -1$.

As in the Remark for part (a), we could justify the above in more detail here using the fact that $\int_1^\infty x^{\alpha p}dx = \lim_{n\to\infty} \int_1^n x^{\alpha p}dx$ by MON, and the fact that $x \mapsto x^{\alpha p}$ is bounded and continuous on $[1,n]$ for all n.