44. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $F_n \subset X$ with $F_n \in \mathcal{M}$ and $\mu(F_n) < \infty$, $\forall n \in \mathbb{N}$. Suppose also that $\mathcal{D} \subset \mathcal{M}$ is a π-system in X with $F_n \in \mathcal{D}$ for all $n \in \mathbb{N}$, and ν is a measure on $(X, \mathcal{M})$ such that $\nu(A) = \mu(A)$ for all $A \in \mathcal{D}$. For $n \in \mathbb{N}$ set $E_n := \cup_{j=1}^n F_j$.

    (a) Use the inclusion-exclusion formula from Question 39 to show for all $n \in \mathbb{N}, A \in \mathcal{D}$ that

    $$\mu(E_n) = \nu(E_n); \qquad \mu(A \cap E_n) = \nu(A \cap E_n).$$

    (b) Now suppose moreover that $\cup_{n=1}^\infty F_n = X$. Show that $\mu(A) = \nu(A)$ for all $A \in \sigma(\mathcal{D})$.

(a) Since $\mathcal{D}$ is a π-system, any intersection of finitely many sets in $\mathcal{D}$ is also in $\mathcal{D}$. In particular $\cap_{j \in J} F_j \in \mathcal{D}$ for all $J \in S(n)$. Therefore using inclusion-exclusion,

$$\mu(E_n) = \sum_{J \in S(n)} (-1)^{|J|+1} \mu(\cap_{j \in J} F_j) = \sum_{J \in S(n)} (-1)^{|J|+1} \nu(\cap_{j \in J} F_j) = \nu(E_n).$$

Similarly, for $A \in \mathcal{D}$ we have $\cap_{j \in J}(A \cap F_j) \in \mathcal{D}$ for all $J \in S(n)$, so

$$\begin{aligned} \mu(A \cap E_n) = \mu(\cup_{j=1}^n (A \cap F_j)) &= \sum_{J \in S(n)} (-1)^{|J|+1} \mu(\cap_{j \in J}(A \cap F_j)) \\ &= \sum_{J \in S(n)} (-1)^{|J|+1} \nu(\cap_{j \in J}(A \cap F_j)) = \nu(A \cap E_n). \end{aligned}$$

(b) For $n \geq N$ and $A \in \mathcal{M}$ define

$$\mu_n(A) = \mu(A \cap E_n); \quad \nu_n(A) = \nu(A \cap E_n).$$

Then μ_n is a measure on $(X, \mathcal{M})$ since for $A_1, A_2, \ldots$ pairwise disjoint in $\mathcal{M}$ we have

$$\mu_n(\cup_{i=1}^\infty A_i) = \mu((\cup_{i=1}^\infty A_i) \cap E_n) = \mu(\cup_{i=1}^\infty (A_i \cap E_n)) = \sum_{i=1}^\infty \mu_n(A_i).$$

Similarly ν_n is a measure on $(X, \mathcal{M})$. Also $\mu_n(X) = \mu(X \cap E_n) = \mu(E_n) \leq \sum_{i=1}^n \mu(F_i) < \infty$.

For any $A \in \mathcal{D}$, by (a) we have $\mu(A \cap E_n) = \nu(A \cap E_n)$ and $\mu(E_n) = \nu(E_n)$. Hence $\mu_n(A) = \nu_n(A)$ for all $A \in \mathcal{D}$, and $\mu_n(X) = \mu(E_n) = \nu(E_n) = \nu_n(X) < \infty$. Hence we can apply Lemma 5.6 (uniqueness lemma for finite measures) to deduce that $\mu_n(A) = \nu_n(A)$ for all $A \in \sigma(\mathcal{D})$. Thus for all $A \in \sigma(\mathcal{D})$ we have

$$\mu(A \cap E_n) = \mu_n(A) = \nu_n(A) = \nu(A \cap E_n).$$

Finally, since $(A \cap E_n) \subset (A \cap E_{n+1})$ for all $n \geq N$ and $\cup_{n=1}^\infty (A \cap E_n) = A \cap (\cup_{n=1}^\infty E_n) = A \cap X = A$, using upward continuity we obtain for all $A \in \sigma(\mathcal{D})$ that

$$\mu(A) = \lim_{n \to \infty} \mu(A \cap E_n) = \lim_{n \to \infty} \nu(A \cap E_n) = \nu(A).$$

45. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. Let $f : \Omega \to [0, \infty]$ be measurable, i.e. f is a nonnegative random variable. For $t \geq 0$ define $L(t) := \int_\Omega e^{-tf(\omega)}\mu(d\omega)$ (the *Laplace transform* of f).

   (a) Show that $\lim_{t\to\infty} L(t) = \mu(\{\omega \in \Omega : f(\omega) = 0\})$. Here we make the convention that $e^{-\infty} = 0$.

   (b) Show that $\lim_{t\downarrow 0} L(t) = \mu(\{\omega \in \Omega : f(\omega) < \infty\})$.

   (c) Show that $\lim_{t\downarrow 0}(t^{-1}(L(0) - L(t))) = \int f d\mu$ if the integral on the right is finite . [Hint: use the fact that $1 - e^{-x} \leq x$ for $x \geq 0$].

   What about if the integral is infinite?

   (a) Fix a sequence $t_n \uparrow \infty$. Let $\omega \in \Omega$. Then

$$\lim_{n\to\infty} e^{-t_n f(\omega)} = \begin{cases} 1 & \text{if } f(\omega) = 0; \\ 0 & \text{if } f(\omega) > 0. \end{cases}$$

Since $t_n > 0$ for all n, we have the domination $|e^{-t_n f(\omega)}| \leq 1$, and here $1 \in L^1(\mu)$, since $\int_\Omega 1\, d\mu = \mu(\Omega) = 1 < \infty$. Therefore, by the Dominated Convergence Theorem

$$L(t_n) = \int_\Omega e^{-t_n f}\, d\mu \overset{n\to\infty}{\longrightarrow} \int_\Omega \mathbf{1}_C\, d\mu = \mu(C),$$

where $C = \{\omega \in \Omega : f(\omega) = 0\}$. Since this convergence holds for any choice of t_n with $t_n \uparrow \infty$, it follows that $L(t) \to \mu(C)$ as $t \to \infty$, as required.

[Here we are using: if $F : [0, \infty) \to \mathbb{R}$ is a function and $a \in \mathbb{R}$ are such that $F(t_n) \to a$ for any sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \uparrow \infty$ as $n \to \infty$, then $F(t) \to a$ as $t \to \infty$.]

(b) Fix a sequence $t_n \downarrow 0$. We have

$$\lim_{n\to\infty} e^{-t_n f(\omega)} = \begin{cases} 1 & \text{if } f(\omega) < \infty; \\ 0 & \text{if } f(\omega) = \infty. \end{cases}$$

As in Part (a), since $t_n > 0$ for all n, we have the domination $|e^{-t_n f(\omega)}| \leq 1$, and here $1 \in L^1(\mu)$, since $\int_\Omega 1\, d\mu = \mu(\Omega) = 1 < \infty$. Therefore, by the Dominated Convergence Theorem

$$L(t_n) = \int_\Omega e^{-t_n f}\, d\mu \overset{n\to\infty}{\longrightarrow} \int_\Omega \mathbf{1}_D\, d\mu = \mu(D),$$

where $D = \{\omega \in \Omega : f(\omega) < +\infty\}$. Since our choice of t_n satisfying $t_n \downarrow 0$ is arbitrary, it follows that $L(t) \to \mu(D)$ as $t \downarrow 0$, as required.

[Here we are using: if $F : (0, \infty) \to \mathbb{R}$ is a function and $a \in \mathbb{R}$ with $F(t_n) \to a$ for any sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \downarrow 0$ as $n \downarrow 0$, then $F(t) \to a$ as $t \downarrow 0$. This is similar to the fact that if a function is sequentially continuous at 0, then it is continuous at 0, which you should have seen in first year Analysis.]

(c) Let us take a sequence $t_n \downarrow 0$. Then

$$t_n^{-1}(L(0) - L(t_n)) = \int_\Omega t_n^{-1}(1 - e^{-t_n f(\omega)})\mu(d\omega) = \int g_n d\mu$$

where we set $g_n(\omega) := t_n^{-1}(1 - e^{-t_n f(\omega)})$. Then $g_n(\omega) \to f(\omega)$ as $n \to \infty$, and also (using the hint) $g_n(\omega) \leq f(\omega)$. Therefore if $\int f d\mu < \infty$ we can use Dominated convergence with dominating function f to deduce that

$$\lim_{n\to\infty} t_n^{-1}(L(0) - L(t_n)) = \int_\Omega f(\omega)\mu(d\omega) = \int f d\mu.$$

If $\int f d\mu = \infty$ we can no longer use Dominated Convergence (DOM). However, in fact we have $0 \leq g_n(\omega) \leq g_{n+1}(\omega)$ for all ω, so we can use Monotone Convergence (MON) to deduce that in this case $\lim_{n\to\infty} t_n^{-1}(L(0) - L(t_n)) = \int f d\mu = \infty$. Thus in all cases $t_n^{-1}(L(0) - L(t_n)) \to \int f d\mu$ as $n \to \infty$. Since this holds for any sequence $t_n \downarrow 0$, it follows that $t^{-1}(L(0) - L(t_n)) \to \int f d\mu$ as $t \downarrow 0$.

46. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Show the following.

    (a) If $f : X \to [-\infty, \infty]$ is measurable, $E \in \mathcal{M}$, $\int_E |f| \, d\mu = 0$, then $f = 0$ a.e. on E.

    (b) If $f \in L^1(\mu)$ with $\int_E f \, d\mu = 0$ for all $E \in \mathcal{M}$, then $f = 0$ a.e. on X.

    (c) If $f \in L^1(\mu)$ with $\left|\int_X f \, d\mu\right| = \int_X |f| \, d\mu$, then either $f \geq 0$ a.e. on X, or $f \leq 0$ a.e. on X.

    (d) If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are measurable functions, then $\{x \in X : f(x) \neq g(x)\} \in \mathcal{M}$.

(a) By the assumption given, setting $g = |f|\mathbf{1}_E$ we have $\int g d\mu = 0$. Also $g \geq 0$ pointwise. Therefore by Question 33 (b) we have $\mu(g^{-1}((0, \infty])) = 0$, so that $\mu(\{x \in E : f(x) \neq 0\} = 0$, or in other words $f = 0$ μ-a.e. on E.

(b) Suppose $f \in L^1(\mu)$ with $\int_E f d\mu = 0$ for all $E \in \mathcal{M}$.

Take $E = \{x \in X : f(x) \geq 0\}$. Then $f\mathbf{1}_E \geq 0$ pointwise and by the stated condition $\int f\mathbf{1}_E d\mu = \int_E f = 0$. Hence by Question 33 (b), $\mu(\{x \in X : f(x)\mathbf{1}_E(x) > 0\}) = 0$, that is, $\mu(f^{-1}((0, \infty])) = 0$. Also setting $g = -f$, we have $\int_E g d\mu = -\int_E f d\mu = 0$ for all E, so by the same argument as above we have $\mu(g^{-1}((0, \infty])) = 0$, that is, $\mu(f^{-1}([-\infty, 0))) = 0$. Therefore

$$\mu(\{x \in X : x \neq 0\}) = \mu(f^{-1}((0, \infty]) \cup f^{-1}([-\infty, 0))) \leq \mu(f^{-1}((0, \infty])) + \mu(f^{-1}([-\infty, 0))) = 0,$$

so $\mu(\{x \in X : x \neq 0\}) = 0$, or in other words $f = 0$ a.e.

(c) Suppose $f \in L^1(\mu)$ with $\left|\int_X f \, d\mu\right| = \int_X |f| \, d\mu$. Set $I^+ = \int f^+ d\mu$ and $I^- = \int f^- d\mu$. Then $I^+ \geq 0$, $I^- \geq 0$ and since $|f| = f^+ - f^-$, our assumption tells us that

$$|I^+ - I^-| = I^+ + I^-$$

which fails unless either $I^+ = 0$ or $I^- = 0$ (or both). But if $I^+ = 0$ then by Question 33 (b) we have $f^+ = 0$ μ-almost everywhere, or in other words $f \leq 0$ μ-a.e. Similarly, if $I^- = 0$ then by Question 33 (b) we have $f^- = 0$ μ-almost everywhere, or in other words $f \geq 0$ μ-a.e.

(d) By Corollary 10.8 and Theorem 10.13 the function $f - g$ is measurable, so by Theorem 10.5 the set $\{x \in X : f(x) \neq g(x)\} = (f - g)^{-1}(\mathbb{R} \setminus \{0\})$ is in $\mathcal{M}$.

47. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable. Suppose $\{h_n\}_{n\geq 1}$ is a sequence in $\mathbb{R}$ such that $h_n \to 0$.

    (a) Show that for any $K \in (0, \infty)$ we have $\int_{-K}^{K} |f(x + h_n) - f(x)|dx \to 0$ as $n \to \infty$. [Hint: first suppose f is continuous, and recall that any continuous real-valued function on a compact interval is bounded.]

    (b) Show that $\int_{-\infty}^{\infty} |f(x + h_n) - f(x)|dx \to 0$ as $n \to \infty$.

    (a) First assume f is continuous. In that case, setting $g_n(x) = |f(x + h_n) - f(x)|$ we have $g_n \to 0$ pointwise as $n \to \infty$, and (assuming n is large enough so that $|h_n| \leq 1$) $|g_n(x)| \leq |f_n(x) + f_n(x + h_n)| \leq 2M$, where we set $M = \sup_{-K-1\leq x\leq K+1} |f(x)|$ which is finite by the hint. Since $\int_{-K}^{K}(2M)dx = 4KM < \infty$, by DOM we have

    $$\int_{-K}^{K} |f(x + h_n) - f(x)|dx = \int_{-K}^{K} g_n(x)dx \to 0 \text{ as } n \to \infty.$$

    In the general case (f maybe not continuous) we have to use Question 43. Given $\varepsilon > 0$, take $w : \mathbb{R} \to \mathbb{R}$ such that w is continuous and $\int_{-\infty}^{\infty} |w(x) - f(x)|dx < \varepsilon/3$, and set $w_n(x) = w(x + h_n)$. Then $\int_{-K}^{K} |w_n(x) - w(x)|dx \to 0$ as $n \to \infty$ by the argument just given. Also, for all n, using Question 37 we have

    $$\int_{-\infty}^{\infty} |w_n(x) - f_n(x)|dx = \int_{-\infty}^{\infty} |w(x + h_n) - f(x + h_n)|dx = \int_{-\infty}^{\infty} |w(x) - f(x)|dx < \varepsilon/3,$$

    so for large enough n, setting $f_n(x) = f(x + h_n)$ we have

    $$\int_{-K}^{K} |f_n - f|dx \leq \int_{-K}^{K} |f_n - w_n|dx + \int_{-K}^{K} |w_n - w|dx + \int_{-K}^{K} |w - f|dx < \varepsilon.$$

    (b) Let $\varepsilon > 0$ and choose K such that $\int_{\mathbb{R}\setminus[-K,K]} |f(x)|dx < \varepsilon$. This can be done by the solution to Question 41. Set $f_n(x) = f(x + h_n)$. Then using (b), choose N so that $\int_{-(K+1)}^{K+1} |f_n(x) - f(x)|dx < \varepsilon$ for all $n \geq N$, and also $|h_n| \leq 1$ for all $n \geq N$. Then for $n \geq N$, we have

    $$\begin{aligned}\int_{-\infty}^{\infty} |f_n(x) - f(x)|dx &\leq \int_{[-K-1,K+1]^c} |f_n(x) - f(x)|dx + \int_{-K-1}^{K+1} |f_n(x) - f(x)|dx \\ &\leq \int_{[-K-1,K+1]^c} |f(x + h_n)|dx + \int_{[-K-1,K+1]^c} |f(x)|dx + \varepsilon \\ &\leq 2\int_{[-K,K]^c} |f(x)|dx + \varepsilon \leq 3\varepsilon\end{aligned}$$

    which gives the result.

48. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Suppose $f, f_1, f_2, \ldots \in \mathbb{R}(X)$ such that $f_n \uparrow f$ pointwise and moreover $f_n \in L^1(\mu)$ and $\sup_n \int f_n d\mu < \infty$. Show that $f \in L^1(\mu)$ and $\int f_n d\mu \to \int f d\mu$ as $n \to \infty$. (This result is sometimes called *Beppo Levi's theorem.*)

Since $f_n \uparrow f$ (pointwise) we have $f_n^+ \uparrow f^+$ and $f_n^- \downarrow f^-$ (pointwise). Basically this is because the function $x \mapsto tox^+$ from $\mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing, whle the function $x \mapsto x^-$ is continuous and nonincreasing.

Hence $f_n^- \leq f_1^-$ pointwise so $\int f_n^- d\mu \leq \int f_1^- d\mu$ which is finite because $f_1 \in L^1(\mu)$.

By assumption there exists $K \in \mathbb{N}$ such that for all n we have $\int f_n^+ d\mu - \int f_n^- d\mu = \int f_n d\mu \leq K$. Therefore for all n we have

$$\int f_n^+ d\mu \leq K + \int f_n^- d\mu \leq K + \int f_1^- d\mu < \infty.$$

Since $0 \leq f_n^+ \uparrow f^+$, by MON we have

$$\int f^+ d\mu = \lim_{n\to\infty} \int f_n^+ d\mu \leq K + \int f_1^- d\mu < \infty.$$

Also $0 \leq f_n^- \leq f_1^-$ pointwise and $\int f_1^- d\mu < \infty$ and $f_n^- \to f^-$ pointwise so we can apply DOM (with dominating function f_1^-) to deduce that

$$\int f^- d\mu = \lim_{n\to\infty} \int f_n^- d\mu \leq \int f_1 d\mu < \infty.$$

Therefore both f^+ and f^- are integrable; hence so is f. Moreover by the Algebra of Limits theorem

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu = \lim_{n\to\infty} \int f_n^+ d\mu - \lim_{n\to\infty} \int f_n^- d\mu = \lim_{n\to\infty} (\int f_n^+ d\mu - \int f_n^- d\mu) = \lim_{n\to\infty} \int f_n d\mu$$

as required.