37. (a) Suppose $g : \mathbb{R} \to \mathbb{R}$ is integrable and $t \in \mathbb{R}$. Show that $\int_{-\infty}^{\infty} g(x-t)dx = \int_{-\infty}^{\infty} g(x)dx$.

(b) Deduce that for any $a, b \in \mathbb{R}$ with $a < b$, $\int_{a+t}^{b+t} g(x-t)dx = \int_a^b g(x)dx$.

(a) Set $h(x) = g(x-t)$. We need to show $\int_{\mathbb{R}} hd\lambda_1 = \int_{\mathbb{R}} gd\lambda_1$, where λ_1 is Lebesgue measure.

First suppose g is nonnegative and simple. By Lemma 11.7(a) we can write $g = \sum_{i=1}^n \alpha_i \mathbf{1}_{A_i}$ with all of the $\alpha_i \geq 0$ and $A_i \in \mathcal{B}$. For all $x \in \mathbb{R}$, note that $x - t \in A \Leftrightarrow x \in A + t$, and hence $\mathbf{1}_A(x-t) = \mathbf{1}_{A+t}(x)$. Hence $h = \sum_{i=1}^n \alpha_i \mathbf{1}_{A_i+t}$. Then using Lemma 11.7(b) and also the translation invariance of λ_1 (Theorem 6.8) we have

$$\int_{\mathbb{R}} hd\lambda_1 = \sum_{i=1}^n \alpha_i \lambda_1(A_i + t) = \sum_{i=1}^n \alpha_i \lambda_1(A_i) = \int_{\mathbb{R}} gd\lambda_1.$$

Now suppose g is nonnegative. Let $(g_n)_{n\geq 1}$ be a sequence of simple functions with $0 \leq g_n \uparrow g$ pointwise (see Theorem 10.12). Set $h_n(x) = g_n(x+t)$ for $x \in \mathbb{R}$. Then h_n is simple and $0 \leq h_n \uparrow h$ pointwise, so by MON and the previous case,

$$\int_{\mathbb{R}} hd\lambda_1 = \lim_{n\to\infty} \int_{\mathbb{R}} h_n d\lambda_1 = \lim_{n\to\infty} \int g_n d\lambda_1 = \int_{\mathbb{R}} gd\lambda_1.$$

For general integrable g we have that $h^+(x) = g^+(x-t)$ for all x, and $h^-(x-t) = g^-(x-t)$ for all x. Hence by the previous case $\int h^+ d\lambda_1 = \int g^+ d\lambda_1$ and $\int h^- d\lambda_1 = \int g^- d\lambda_1$. Hence

$$\int hd\lambda_1 = \int h^+ d\lambda_1 - \int h^- d\lambda_1 = \int g^+ d\lambda_1 - \int g^- d\lambda_1 = \int gd\lambda_1.$$

(b) Set $f(x) := g(x)\mathbf{1}_{(a,b)}(x)$. Then $f(x-t) = g(x-t)\mathbf{1}_{(a+t,b+t)}(x)$. Hence by part (a),

$$\int_{a+t}^{b+t} g(x-t)dx = \int_{-\infty}^{\infty} g(x-t)\mathbf{1}_{(a+t,b+t)}(x)dx = \int_{-\infty}^{\infty} f(x-t)dx = \int_{-\infty}^{\infty} f(x)dx = \int_a^b g(x)dx.$$

38. Let μ be counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$.

(a) Let $k \in \mathbb{N}$. Show that if $f : X \to [0, \infty)$ with $f(n) = 0$ for all $n > k$, then $\int_{\mathbb{N}} fd\mu = \sum_{i=1}^k f(i)$.

(a) Under the given assumption, f is simple and nonnegative. Indeed $f = \sum_{i=1}^k f(i)\mathbf{1}_{\{i\}}$ so by Lemma 11.7(b), since μ is counting measure so $\mu(\{i\}) = 1$ for each i, we have

$$\int fd\mu = \sum_{i=1}^k f(i)\mu(\{i\}) = \sum_{i=1}^k f(i).$$

(b) Show that if $g : \mathbb{N} \to [0, \infty)$ then $\int_{\mathbb{N}} gd\mu = \sum_{n=1}^{\infty} g(n)$.

(b) For $n \in \mathbb{N}$, define $g_n(i) = g(i)$ for $i \leq n$, with $g_n(i) = 0$ for $i \geq n$. Then by part (a) $\int g_n d\mu = \sum_{i=1}^n g(i)$.

Since we assume $g \geq 0$ we have $g_n \uparrow g$ pointwise and so by MON,

$$\int gd\mu = \lim_{n\to\infty} \int g_n d\mu = \lim_{n\to\infty} \sum_{i=1}^n g(i) = \sum_{i=1}^{\infty} g(i).$$

(c) Suppose $h : \mathbb{N} \to \mathbb{R}$ with $\sum_{n=1}^{\infty} |h(n)| < \infty$. Show that $\int_{\mathbb{N}} h d\mu = \sum_{i=1}^{\infty} h(i)$.

Since $h^+ \leq |h|$ and $h^- \leq |h|$ pointwise we have $\sum_n h^+(n) \leq \sum_n |h(n)| < \infty$ and $\sum_n h^-(n) < \infty$. Hence by (b) both h^+ and h^- are in $L^1(\mu)$, and

$$\begin{aligned}\int h d\mu = \int h^+ d\mu - \int h^- d\mu &= \Big(\sum_{n=1}^{\infty} h^+(n)\Big) - \sum_{n=1}^{\infty} h^-(n) \\ &= \lim_{k\to\infty} \Big(\sum_{n=1}^{k} h^+(n)\Big) - \lim_{k\to\infty} \sum_{n=1}^{k} h^+(n) \\ &= \lim_{k\to\infty} \sum_{n=1}^{k} (h^+(n) - h^-(n)) = \lim_{k\to\infty} \sum_{n=1}^{k} h(n) = \sum_{n=1}^{\infty} h(n),\end{aligned}$$

where we used the definition of an infinite sum in the second line, and the algebra of limits theorem at the start of the third line.

39. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Suppose $F_1, \ldots, F_n$ are subsets of X with $F_i \in \mathcal{M}$ and $\mu(F_i) < \infty$ for each $i \in [n]$, where we set $[n] := \{1, \ldots, n\}$. For $S \subset [n]$ let $|S|$ denote the number of elements of S. Use the linearity of integration, and the fact that $\mu(A) = \int_X \mathbf{1}_A$ for any $A \in \mathcal{M}$, to prove the inclusion-exclusion formula from Question 44, namely

$$\mu(\cup_{i=1}^n F_i) = \sum_{J \in S(n)} (-1)^{|J|+1} \mu(\cap_{j \in J} F_j), \qquad \text{where} \quad S(n) := \{J \subset \{1, \ldots, n\} : J \neq \varnothing\}.$$

[Hint: for any sets $G_1, \ldots, G_k \in \mathcal{M}$ we have $\mathbf{1}_{\cap_{i=1}^k G_i} = \prod_{i=1}^k \mathbf{1}_{G_i}$.]

By Lemma 11.7 (integration of simple functions formula), $\mu(\cup_{i=1}^n F_i) = \int \mathbf{1}_{\cup_{i=1}^n F_i} d\mu$. By the hint

$$\mathbf{1}_{\cup_{i=1}^n F_i} = 1 - \mathbf{1}_{\cap_{i=1}^n F_i^c} = 1 - \prod_{i=1}^n \mathbf{1}_{F_i^c} = 1 - \prod_{i=1}^n (1 - \mathbf{1}_{F_i}).$$

By a binomial-type expansion, for any real $x_1, \ldots, x_n$ we have

$$\prod_{i=1}^n (1 - x_i) = (1 - x_1)(1 - x_2) \cdots (1 - x_n) = 1 + \sum_{J \in S(n)} \prod_{i \in J} (-x_i) = 1 + \sum_{J \in S(n)} \prod_{i \in J} (-1)^{|J|} \prod_{i \in J} x_i$$

Taking $x_i = \mathbf{1}_{F_i}$ and using the hint again, we obtain that

$$\mathbf{1}_{\cup_{i=1}^n F_i} = 1 - \left(1 + \sum_{J \in S(n)} (-1)^{|J|} \prod_{i \in J} \mathbf{1}_{F_i}\right) = \sum_{J \in S(n)} (-1)^{|J|+1} \mathbf{1}_{\cap_{i \in J} F_i}.$$

Using the linearity of integration we obtain that

$$\mu(\cup_{i=1}^n F_i) = \int \mathbf{1}_{\cup_{i=1}^n F_i} d\mu = \sum_{J \in S(n)} (-1)^{|J|+1} \int \mathbf{1}_{\cap_{i \in J} F_i} d\mu = \sum_{J \in S(n)} (-1)^{|J|+1} \mu(\cap_{i \in J} F_i).$$

40. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Suppose $f, g, h \in L^1(\mu)$.

 (a) For $F \in L^1(\mu)$ set $\|F\|_1 := \int |f| d\mu$. Show that $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$.

 (b) Show that $f - h \in L^1(\mu)$ and $h - g \in L^1(\mu)$ and $\|f - g\|_1 \leq \|f - h\|_1 + \|h - g\|_1$.

 (a) By the triangle inequality, for all $x \in X$ we have $|f(x)+g(x)| \leq |f(x)|+|g(x)|$, so $0 \leq |f+g| \leq |f| + |g|$ pointwise. By Lemma 11.5(a), and then linearity of integration,

$$\|f + g\|_1 = \int |f + g| d\mu \leq \int (|f| + |g|) d\mu = \int |f| d\mu + \int |g| d\mu = \|f\|_1 + \|g\|_1.$$

 (b) Since $| - h(x)| = |h(x)|$ for all $x \in X$, we have $\| - h\|_1 = \int | - h| d\mu = \int |h| d\mu = \|h\|_1$. Hence by part (a) $\|f - h\|_1 \leq \|f\|_1 + \| - h\|_1 = \|f\|_1 + \|h\|_1 < \infty$. Thus $f - h \in L^(\mu)$ and similarly $h - g \in L^1(\mu)$.

 Since $f - g = (f - h) + (h - g)$ we have by part (a) that $\|f - g\|_1 \leq \|f - h\|_1 + \|h - g\|_1$.

41. Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable. Show that there exists integrable $g : \mathbb{R} \to \mathbb{R}$ such that $\int_{-\infty}^{\infty} |f(x) - g(x)| dx < \varepsilon$, and g has bounded support (i.e., there exists $n \in \mathbb{N}$ with $g(x) = 0$ whenever $|x| > n$).

 Setting $f_n := |f| \mathbf{1}_{(-n,n)}$ we have $f_n \uparrow |f|$ pointwise so by MON, we have as $n \to \infty$ that

$$\int_{-n}^{n} f(x) dx = \int f_n d\lambda_1 \to \int |f| d\lambda_1 < \infty$$

 so we can choose N such that $\int_{-N}^{N} |f(x)| dx > \int_{-\infty}^{\infty} |f(x)| dx - \varepsilon$.

 Take $g = f \mathbf{1}_{(-N,N)}$. Then g has bounded support and

$$\int_{-\infty}^{\infty} |f(x) - g(x)| dx = \int_{-\infty}^{N} |f(x)| dx + \int_{N}^{\infty} |f(x)| dx = \int_{-\infty}^{\infty} |f(x)| dx - \int_{-N}^{N} |f(x)| dx < \varepsilon.$$

42. A function $g : \mathbb{R} \to \mathbb{R}$ is called a **step function** if we can write $g = \sum_{i=1}^{k} c_i \mathbf{1}_{I_i}$ for some $k \in \mathbb{N}$, $(c_1, \ldots, c_k) \in \mathbb{R}^k$ and $I_1, \ldots, I_k$ intervals in $\mathbb{R}$.

 Suppose $f : \mathbb{R} \to [0, \infty)$ is simple and has bounded support. Let $\varepsilon > 0$. Show that there exists a step function $g : \mathbb{R} \to \mathbb{R}$ such that $\int_{-\infty}^{\infty} |g - f| dx < \varepsilon$. *Hint: Recall Questions 17 and 23.*

 First assume $f = \mathbf{1}_A$ for some bounded Borel set A. By Question 17, given $\varepsilon > 0$ we can find a set U which is a finite union of bounded intervals such that $\lambda(A \triangle U) < \varepsilon/2$. Also we can take these intervals to be half-open, and then since $\mathcal{U}$ is an algebra by Qusestion 23, the set U is in $\mathcal{U}$ and therefore is in fact a finite union of *pairwise disjoint* half-open bounded intervals, denoted $I_1, \ldots, I_k$ say.

 Clearly $\mathbf{1}_U = \sum_{i=1}^{k} \mathbf{1}_{I_i}$ is a step function, and since $|\mathbf{1}_A(x) - \mathbf{1}_U(x)| = \mathbf{1}_{A \triangle U}(x)$ for all $x \in \mathbb{R}$ we have $\int_{-\infty}^{\infty} |\mathbf{1}_A(x) - \mathbf{1}_U(x)| dx = \int \mathbf{1}_{A \triangle U}(x) dx = \lambda_1(A \triangle U) < \varepsilon$.

 Now suppose f is simple, $f \geq 0$. By Lemma 11.7 we can write $f = \sum_{i=1}^{\ell} a_i \mathbf{1}_{A_i}$ with A_i all bounded and measurable, and $a_i \geq 0$ for all i. Assume the a_i are not all zero (otherwise $f \equiv 0$ which is itself a step function). Then by the case considered earlier we can find step functions $h_1, \ldots, h_\ell$

such that $\int |h_i - \mathbf{1}_{A_i}| < \varepsilon/(\ell \max(a_1, \dots, a_\ell))$ for each i. Setting $h = \sum_{i=1}^{\ell} a_i h_i$ gives us a step function with

$$\int |f-h| d\lambda_1 = \int \left| \sum a_i (\mathbf{1}_{A_i} - h_i) \right| d\lambda_1 \le \sum_{i=1}^{\ell} a_i \int |\mathbf{1}_{A_i} - h_i| d\lambda_1 < \varepsilon.$$

43. Suppose $f : \mathbb{R} \to \mathbb{R}$ is in L^1. Let $\varepsilon > 0$. Using Question 42, show there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $\|f - g\|_1 < \varepsilon$, i.e. $\int_{-\infty}^{\infty} |f(x) - g(x)| dx < \varepsilon$.

First suppose $f = \mathbf{1}_I$ for I an interval with left endpoint a and right endpoint b. Take $f_n(x) = f(x)$ for $x \in I$ and for $x \le a - 1/n$, and for $x \ge b + 1/n$, with the value of f_n interpolated linearly between $x = a - 1/n$ and $x = a$, and the value of f_n interpolated linearly between $x = b$ and $x = b + 1/n$.

Then f_n is continuous and $|f_n - f| \le \mathbf{1}_{[a-1/n,a] \cup [b,b+1/n]}$ so that $\int_{-\infty}^{\infty} |f_n - f| dx \le 2/n$, so $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

Now take a new function f. Assume $f : \mathbb{R} \to [0, \infty)$ is integrable. Using Question 41, take $g \in L^1$ with bounded support and $\|g - f\|_1 \le \varepsilon/9$ and $g \ge 0$ (the solution to Question 41 shows that if $f \ge 0$ we can take $g \ge 0$).

Now take g_n simple with $0 \le g_n \le g$ pointwise and $g_n \uparrow g$ pointwise. Then by MON, we have $\int g_n d\mu \uparrow \int g d\mu$, so given $\varepsilon > 0$ we can choose N such that $\int |g - g_N| \le \varepsilon/9$, i.e. $\|g - g_N\|_1 < \varepsilon/9$. Note also that g_N has bounded support since $0 \le g_N \le g$ pointwise and g has bounded support.

Since g_N is simple, we can and do write $g_N = \sum_{i=1}^{\ell} a_i \mathbf{1}_{A_i}$ with A_i all bounded and measurable. Assume the a_i are not all zero (otherwise $g = 0$ which is continuous). By the case we started with in this solution, we can find continuous functions $h_1, \dots, h_\ell$ with $\int |h_i - \mathbf{1}_{A_i}| \le \varepsilon/(9\ell \max(|a_1|, \dots, |a_\ell|))$ for each i. Then setting $h = \sum_{i=1}^{\ell} a_i h_i$ gives us a continous function with

$$\int |g_N - h| d\lambda_1 = \int \sum a_i (\mathbf{1}_{A_i} - h_i) d\lambda_1 \le \sum_{i=1}^{\ell} a_i \int |\mathbf{1}_{A_i} - h_i| d\lambda_1 \le \varepsilon/9.$$

Then using Question 40 we obtain that $\|f - h\|_1 \le \|f - g\| + \|g - g_N\|_1 + \|g_N - h\|_1 \le \varepsilon/3$.

Finally if $f : \mathbb{R} \to \mathbb{R} \in L^1$, by the preceding argument we can find continuous functions F_1, F_2 in L^1 with $\|F_1 - f^+\|_1 \le \varepsilon/3$ and $\|F_2 - f^-\|_1 \le \varepsilon/3$. Then by Question 40 again $F_1 - F_2$ is in L^1 (and is continuous) with $\|f - (F_1 - F_2)\|_1 = \|f^+ - F_1\|_1 + \|f^- - F_2\|_+ \le \varepsilon$.