31. (a) Let $(X, \mathcal{M})$ be a measurable space, and let $f_n : X \to \mathbb{R}$ be measurable functions. Show that the set of points
$$\{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists in } \mathbb{R}\}$$
is in $\mathcal{M}$.
(b) Taking $(\Omega, \mathcal{F}, \mathbb{P})$ to be a probability space, and random variables (i.e., measurable functions) $Y_1, Y_2, \ldots : \Omega \to \mathbb{R}$ show that for any constant $\mu \in \mathbb{R}$ the set:
$$\left\{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Y_i(\omega) = \mu\right\}$$
is in $\mathcal{F}$. Deduce that expressions like $\mathbb{P}[\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Y_i = \mu]$ are meaningful.

(a) The complement of the set in question is:

$$\begin{aligned}\{x \in X : \liminf_{n\to\infty} f_n(x) = +\infty\} &\cup \{x \in X : \limsup_{n\to\infty} f_n(x) = -\infty\}\\ &\cup \{x \in X : \liminf_{n\to\infty} f_n(x) < \limsup_{n\to\infty} f_n(x)\}\\ &=: A_+ \cup A_- \cup B.\end{aligned}$$

Hence it is enough to show that each of the sets A_+, A_-, B is in $\mathcal{M}$.

We have
$$A_+ = \bigcap_{k=1}^{\infty} \left\{x \in X : \liminf_{n\to\infty} f_n(x) > k\right\}.$$
We have shown in Thm 9.12 that $\liminf_{n\to\infty} f_n$ is measurable, hence each set of the intersection above is in $\mathcal{M}$. Therefore, $A_+ \in \mathcal{M}$. By a similar argument, $A_- \in \mathcal{M}$.

We write B as

$$\begin{aligned}B &= \textstyle\bigcup_{r\in\mathbb{Q}} \{x \in X : \liminf_{n\to\infty} f_n(x) < r < \limsup_{n\to\infty} f_n(x)\}\\ &= \textstyle\bigcup_{r\in\mathbb{Q}} (\{x \in X : \liminf_{n\to\infty} f_n(x) < r\} \cap \{x \in X : r < \limsup_{n\to\infty} f_n(x)\})\end{aligned}$$

(where $\mathbb{Q}$ is the set of rational numbers). This represents B as a countable union of sets that are the intersection of two sets. For each fixed $r \in \mathbb{Q}$, the two sets are in $\mathcal{M}$, since $\liminf_{n\to\infty} f_n$ and $\limsup_{n\to\infty} f_n$ are measurable. Hence $B \in \mathcal{M}$, and the statement is proved.

(b) RVs are measurable functions by definition, so $f_n := \frac{1}{n}\sum_{i=1}^{n} Y_i$, $n = 1, 2, \ldots$ are measurable functions by Theorems 7.2.1 and 7.2.2. We have

$$\{\omega \in \Omega : \lim_{n\to\infty} f_n(\omega) = \mu\} = \{\omega \in \Omega : \limsup_{n\to\infty} f_n(\omega) = \mu\} \cap \{\omega \in \Omega : \liminf_{n\to\infty} f_n(\omega) = \mu\}.$$

Since $\limsup_{n\to\infty} f_n$ is measurable, and the first set is the inverse image of the Borel set $\{\mu\}$ under this function, this set is in $\mathcal{F}$. Similarly, the second set is also in $\mathcal{F}$. Hence the expression $\mathbb{P}[\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Y_i = \mu]$ is meaningful, because $\mathbb{P}$ is defined on the set in question.

32. Let $(X, \mathcal{M})$ be a measurable space.
(a) Show that if $E \in \mathcal{M}$, then its indicator function $\mathbf{1}_E$ defined by $\mathbf{1}_E(x) = 1$ for $x \in E$ and $\mathbf{1}_E(x) = 0$ for $x \notin E$, is a measurable function.
(b) Let $f : X \to \mathbb{R}$ be function with finite range $f(X) = \{\alpha_1, \ldots, \alpha_n\}$ (with $\alpha_1, \ldots, \alpha_n$ distinct), so that $f = \sum_{i=1}^n \alpha_i \mathbf{1}_{A_i}$, where $A_i = \{x \in X : f(x) = \alpha_i\}$. Show that f is measurable if and only if $A_1, \ldots, A_n \in \mathcal{M}$.

(a) Let $f = \mathbf{1}_E$. Then

$$f^{-1}((\alpha, \infty]) = \begin{cases} X & \text{if } \alpha < 0 \\ E & \text{if } 0 \leq \alpha < 1 \\ \varnothing & \text{if } \alpha \geq 1 \end{cases}$$

and since $\varnothing \in \mathcal{M}$ and $X \in \mathcal{M}$, if $E \in \mathcal{M}$ we have for all $\alpha \in \mathbb{R}$ that $f^{-1}((\alpha, \infty]) \in \mathcal{M}$, so f is measurable.

(b) First suppose $A_i \in \mathcal{M}$ for $1 \leq i \leq n$. Then each of the functions $\mathbf{1}_{A_i}$ is measurable by part (a). Therefore $\alpha_i \mathbf{1}_{A_i}$ is also measurable (for each i) by Corollary 9.9. Hence by Theorem 9.10 the function $f = \sum_{i=1}^n \alpha_i \mathbf{1}_{A_i}$ is measurable.

Conversely, suppose $A_i \notin \mathcal{M}$ for some i. Then for this choice of i we have $f^{-1}(\{\alpha_i\}) = A_i \notin \mathcal{M}$, and therefore f is not measurable, since if it were measurable, by Theorem 9.6 we would have $f^{-1}(E)$ measurable for all Borel $E \subset \mathbb{R}$, in particular for $E = \{\alpha_i\}$ (which is a Borel set).

33. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $f : X \to [0, \infty]$ is measurable.
    (a) Prove that if $a \in (0, \infty)$ then $\mu(f^{-1}[a, \infty]) \leq a^{-1} \int f d\mu$.
    (b) Prove that if $\int f d\mu = 0$, then $\mu(f^{-1}((0, \infty])) = 0$.

    (a) Since f is measurable the set $A := f^{-1}([a, \infty]) = \cap_{n=1}^{\infty} f^{-1}((a - 1/n, \infty])$ is in $\mathcal{M}$ (being a countable intersection of sets in $\mathcal{M}$). Alternatively, $A \in \mathcal{M}$ by Theorem 9.6.
    Set $g(x) = a\mathbf{1}_A(x)$ for all $x \in X$. Then $g \leq f$ pointwise since $g(x) = 0$ for $x \notin A$ and $g(x) = a \leq f(x)$ for $x \in A$. Therefore $\int f d\mu \geq \int g d\mu = a\mu(A)$ by Lemmas 10.5a and 10.7b, and hence $\mu(f^{-1}([a, \infty])) = \mu(A) \leq a^{-1} \int f d\mu$ as claimed.
    (b) Now suppose $\int f d\mu = 0$. Then since $f^{-1}((0, \infty]) = \cup_{n=1}^{\infty} f^{-1}([(1/n), \infty])$, by part (a) and countable subadditivity of measure (Theorem 3.3 (iii)) we have

$$\mu(f^{-1}((0, \infty])) \leq \sum_{n=1}^{\infty} \mu(f^{-1}([(1/n), \infty])) \leq \sum_{n=1}^{\infty} n \int f d\mu = 0.$$

34. Let $(X, \mathcal{M})$ be a measurable space. Suppose $f : X \to [0, \infty)$ and $g : X \to [0, \infty)$ are measurable functions. Define the set $A \subset X \times \mathbb{R} \times \mathbb{R}$ by $A := \{(x, s, t) : f(x) > s, g(x) > t\}$. Let $\mathcal{B}$ denote the Borel σ-algebra in $\mathbb{R}$. Show that $A \in \mathcal{M} \otimes \mathcal{B} \otimes \mathcal{B}$.

We have

$$\begin{aligned} A &= \cup_{q,r \in \mathbb{Q}} \{(x, s, t) : f(x) > q > s, g(x) > r > t\} \\ &= \cup_{q,r \in \mathbb{Q}} (f^{-1}((q, \infty)) \cap g^{-1}((r, \infty)) \times (-\infty, q) \times (-\infty, r)) \end{aligned}$$

which is a countable union of sets in $\mathcal{M} \otimes \mathcal{B} \otimes \mathcal{B}$ and hence is itself in $\mathcal{M} \otimes \mathcal{B} \otimes \mathcal{B}$.

35. (a) Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. Show that that for all $A \subset X \times Y$ with $A \in \mathcal{M} \otimes \mathcal{N}$, and all $y \in Y$, the horizontal cross-section $A_{[y]}$ of A defined by

$$A_{[y]} := \{x \in X : (x, y) \in A\}$$

satisfies $A_{[y]} \in \mathcal{M}$.

(b) Suppose $f : X \to [0, \infty]$ is such that $\text{hyp}(f) \in \mathcal{M} \otimes \mathcal{B}$. Show that f is a measurable function.

(a) Fix $y \in Y$. Let $\mathcal{F}$ be the collection of $A \subset X \times Y$ such that $A_{[y]} \in \mathcal{M}$.
We claim that $\mathcal{F}$ is a σ-algebra. Indeed, $\varnothing_{[y]} = \varnothing \in \mathcal{M}$, so $\varnothing \in \mathcal{F}$. Also, if $A \in \mathcal{F}$, then

$$(A^c)_{[y]} = \{x \in X : (x, y) \in A^c\} = \{x \in X : (x, y) \in A\}^c = (A_{[y]})^c \in \mathcal{M},$$

so $A^c \in \mathcal{F}$. Also if $A_n \in \mathcal{F}$ for $n = 1, 2, 3, \ldots$, then setting $A = \cup_{n=1}^\infty A_n$ we have

$$A_{[y]} = \{x \in X : (x, y) \in \cup_{n=1}^\infty A_n\} = \cup_{n=1}^\infty \{x \in X : (x, y) \in A_n\} = \cup_{n=1}^\infty ((A_n)_{[y]}) \in \mathcal{M},$$

so $A \in \mathcal{F}$. Thus we have verified the claim.
We claim also that $\mathcal{R} \subset \mathcal{F}$, where $\mathcal{R}$ is the collection of measurable rectangles in $X \times Y$. Indeed if $A \in \mathcal{M}$ and $B \in \mathcal{N}$ then

$$(A \times B)_{[y]} = \begin{cases} A \text{ if } y \in B \\ \varnothing \text{ otherwise.} \end{cases}$$

By the two preceding claims, $\mathcal{F}$ is a σ-algebra with $\mathcal{R} \subset \mathcal{F}$, and therefore $\mathcal{M} \otimes \mathcal{N} = \sigma(\mathcal{R}) \subset \mathcal{F}$. In other words, every $A \in \mathcal{M} \otimes \mathcal{N}$ is in $\mathcal{F}$, which is what we needed.

(b) Suppose $f : X \to [0, \infty]$ with $\text{hyp}(f) \in \mathcal{M} \otimes \mathcal{B}$. Then for all $y > 0$,

$$f^{-1}((y, \infty]) = \{x \in X : f(x) > y\} = \{x \in X : (x, y) \in \text{hyp}(f)\} = (\text{hyp}(f))_{[y]}$$

which is in $\mathcal{M}$ by part (a). If $y < 0$ then $f^{-1}((y, \infty]) = X \in \mathcal{M}$, and for $y = 0$ we have $f^{-1}((0, \infty]) = \cup_{n=1}^\infty f^{-1}((1/n, \infty]) \in \mathcal{M}$.
Therefore $f^{-1}((y, \infty]) \in \mathcal{M}$ for all $y \in \mathbb{R}$, so f is measurable.

36. Let $W \in \mathcal{B}$ (the Borel sets in $\mathbb{R}$) with $W \neq \varnothing$. Show that $\mathcal{B}_W$ is the σ-algebra (in W) generated by the collection of all sets of the form $(-\infty, a] \cap W$ with $a \in \mathbb{R}$.

[Hint: $\mathcal{B}_W$ was defined in Chapter 9, but also $\mathcal{B}_W = \{A \cap W : A \in \mathcal{B}\}$.]

According to definition 9.3, $\mathcal{B}_W = \{A : A \subset W, A \in \mathcal{B}\}$. This is a σ-algebra in W since $\varnothing \in \mathcal{B}_W$, and if $A \in \mathcal{B}_W$ then $W \setminus A$ is a Borel set contained in W, so $W \setminus A \in \mathcal{B}_W$, and if $A_1, A_2, \cdots$ are in $\mathcal{B}_W$, then $\cup_{i=1}^\infty A_i$ is a Borel set contained in W so $\cup_{i=1}^\infty A_i \in \mathcal{B}_W$.

Let $\mathcal{I}_W = \{(-\infty, a] \cap W : a \in \mathbb{R}\}$, and let $\sigma_W(\mathcal{I}_W)$ be the σ-algebra in W generated by $\mathcal{I}_W$. We are required to show that $\mathcal{B}_W = \sigma_W(\mathcal{I}_W)$.

Clearly $\mathcal{I}_W \subset \mathcal{B}_W$, since for any $a \in \mathbb{R}$ we have $(-\infty, a] \in \mathcal{B}$ so also $(-\infty, a] \cap W \in \mathcal{B}$ (and also $(-\infty, a] \cap W \subset W$). We showed above that $\mathcal{B}_W$ is a σ-algebra in W, and therefore $\sigma_W(\mathcal{I}_W) \subset \mathcal{B}_W$.

It remains to show that $\mathcal{B}_W \subset \sigma_W(\mathcal{I}_W)$. By the hint, every set in $\mathcal{B}_W$ can be written as $A \cap W$ with $A \in \mathcal{B}$ (it is easy to see that this is really true).

Let $\mathcal{F} = \{A \subset \mathbb{R} : A \cap W \in \sigma_W(\mathcal{I}_W)\}$. We claim this is a σ-algebra in $\mathbb{R}$. Indeed, $\varnothing \in \mathcal{F}$, and if $A \in \mathcal{F}$ then $(\mathbb{R} \setminus A) \cap W = W \setminus (A \cap W) \in \sigma_W(\mathcal{I}_W)$, and if $A_1, A_2, \ldots \in \mathcal{F}$ then $(\cup_i A_i) \cap W = \cup_i (A_i \cap W) \in \sigma_W(\mathcal{I}_W)$.

Let $\mathcal{I}_0 = \{(-\infty, a] : a \in \mathbb{R}\}$. By definition, for all $A \in \mathcal{I}_0$ we have $A \cap W \in \mathcal{I}_W \subset \sigma_W(\mathcal{I}_W)$, so $\mathcal{I}_0 \subset \mathcal{F}$. Therefore $\mathcal{B} = \sigma(\mathcal{I}_0) \subset \mathcal{F}$. Hence, for every $A \in \mathcal{B}$ we have $A \cap W \in \sigma_W(\mathcal{I}_W)$, and hence $\mathcal{B}_W \subset \sigma_W(\mathcal{I}_W)$ as required.