26. (a) Show that if $U \subset \mathbb{R}^2$ is open and $x \in U$, then we can find a rectangle $R \in \mathcal{I}_2$ with corners having rational coordinates such that $x \in R \subset U$. *[We say that a set $A \subset \mathbb{R}^2$ is* open *if for every $x \in A$ there is a disk of positive radius centred on x that is contained in A.]*

    (b) Show that $\sigma(\mathcal{O}_2) = \mathcal{B}_2$, where $\mathcal{O}_2$ is the class of all open sets in $\mathbb{R}^2$, and $\mathcal{B}_2$ is the Borel σ-algebra in $\mathbb{R}^2$ (see Definition 9.1).

    (a) Let $U \in \mathcal{O}_2$ and let $x \in U$. Since U is open, there exists $r > 0$ depending on x such that $B(x, r)$ (the closed Euclidean disk centred on x with radius r) is contained in U. Therefore writing $x = (x_1, x_2)$ we have $(x_1 - r/2, x_1 + r/2) \times (x_2 - r/2, x_2 + r/2) \subset B(x, r) \subset U$.

    Now take rational q_1, q_2, r_1, r_2 with $x_1 - r/2 < q_1 < x_1 < r_1 < x_1 + r/2$, and $x_2 - r/2 < q_2 < x_2 < r_2 < x_2 + r/2$. Then

$$x \in (q_1, r_1] \times (q_2, r_2] \subset (x_1 - r/2, x_1 + r/2) \times (x_2 - r/2, x_2 + r/2) \subset B(x, r) \subset U,$$

    so taking $R = (q_1, r_1] \times (q_2, r_2]$ does the trick.

    (b) Recall from Definition 9.1 in the notes that $\mathcal{B}_2 = \sigma(\mathcal{I}_2)$, where $\mathcal{I}_2$ is the collection of all sets of the form $(a, b] \times (c, d]$. Since we can argue as in part (a) for any $x \in U$,

$$U = \cup_{(q_1, r_1, q_2, r_2) \in S}(q_1, r_1] \times (q_2, r_2],$$

    where we define $S := \{(q_1, r_2, q_2, r_2) \in \mathbb{Q}^4 : q_1 < r_1, q_2 < r_2, (q_1, r_1] \times (q_2, r_2] \subset U\}$. Since $\mathbb{Q}^4$ is countable, this shows that U is a countable union of sets in $\mathcal{I}_2$, and therefore is in $\sigma(\mathcal{I}_2) = \mathcal{B}_2$. Thus $\mathcal{O}_2 \subset \mathcal{B}_2$, and since $\mathcal{B}_2$ is a σ-algebra, also $\sigma(\mathcal{O}_2) \subset \mathcal{B}_2$.

    For the inclusion the other way, given $A = (a, b] \times (c, d] \in \mathcal{I}_2$, setting $A_n = (a, b + 1/n) \times (c, d + 1/n)$ we have $A = \cap_{n=1}^{\infty} A_n$ and moreover $A_n \in \mathcal{O}_2$ (i.e., A_n is open) for each n. Thus A is a countable intersection of sets in $\mathcal{O}_2$.

    Since $\sigma(\mathcal{O}_2)$ is a sigma-algebra containing $\mathcal{O}_2$, therefore $A \in \sigma(\mathcal{O}_2)$, so $\mathcal{I}_2 \subset \sigma(\mathcal{O}_2)$, and since $\sigma(\mathcal{O}_2)$ is a σ-algebra, also $\mathcal{B}_2 = \sigma(\mathcal{R}_2) \subset \sigma(\mathcal{O}_2)$.

27. Suppose ρ is a rotation (about the origin) on $\mathbb{R}^2$, i.e. pre-multiplication by a 2×2 matrix M with $M^T = M^{-1}$ (viewing elements of $\mathbb{R}^2$ as column vectors).

    (a) Show that $|\rho(x)| = |x|$ for all $x \in \mathbb{R}^2$, where for $x = (x_1, x_2) \in \mathbb{R}^2$ we put $|x| = \sqrt{x_1^2 + x_2^2}$.
    Consider x as a column vector i.e. $x = (x_1, x_2)'$; then $|x|^2 = x'x$ so $|\rho(x)|^2 = (Mx)'(Mx) = x'M^T Mx = x'x = |x|^2$.

    (b) Show that $\rho(A) \in \mathcal{B}_2$ for all $A \in \mathcal{B}_2$.
    Suppose $U \in \mathcal{O}_2$ (the class of open sets). Then $\rho(U)$ is also open since if $y \in \rho(U)$, then $\rho^{-1}y \in U$ so for some $r > 0$ we have $B(\rho^{-1}y, r) \subset U$ (since U is assumed open), so $B(y, r) = \rho(B(\rho^{-1}y, r)) \subset \rho(U)$. Hence $\rho(U)$ is open, so $\rho(U) \in \mathcal{B}_2$ since $\mathcal{O}_2 \subset \mathcal{B}_2$ by Question 26.
    Let $\mathcal{F}$ be the class of sets $A \subset \mathbb{R}^2$ such that $\rho(A) \in \mathcal{B}_2$. By the preceding paragraph $\mathcal{O}_2 \subset \mathcal{F}$. Also $\mathcal{F}$ is a σ-algebra in $\mathbb{R}^2$ because:

    - $\rho(\varnothing) = \varnothing$ is a Borel set, so $\varnothing \in \mathcal{F}$.
    - If $A \in \mathcal{F}$ then $\rho(\mathbb{R}^2 \setminus A) = \mathbb{R}^2 \setminus \rho(A) \in \mathcal{B}_2$, so $\mathbb{R}^2 \setminus A \in \mathcal{F}$.

- If $A_1, A_2, \ldots \in \mathcal{F}$ then $\rho(\cup_{n=1}^{\infty} A_n) = \cup_{n=1}^{\infty} \rho(A_n) \in \mathcal{F}$, so $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Therefore $\sigma(\mathcal{O}_2) \subset \mathcal{F}$, so by Question 26, $\mathcal{B}_2 \subset \mathcal{F}$ which gives the desired conclusion.

(c) Define a measure μ on $\mathcal{B}_2$ by $\mu(A) = \lambda_2(\rho(A))$ for all $A \in \mathcal{B}_2$. Show that μ is translation invariant.

For $A \in \mathcal{B}_2$ and $x \in \mathbb{R}^2$ we have $\rho(A + x) = \rho(A) + \rho(x)$ (distributive law for matrix multiplication) so

$$\mu(A + x) = \lambda_2(\rho(A + x)) = \lambda_2(\rho(A) + \rho(x)) = \lambda_2(\rho A) = \mu(A),$$

where the penultimate inequality is because λ_2 is translation invariant. Thus μ is translation invariant.

(d) Show that λ_2 is rotation invariant, i.e. $\lambda_2(\rho(A)) = \lambda_2(A)$ for all Borel $A \subset \mathbb{R}^2$ (and for any rotation ρ).

By part (c) along with the fact that every translation-invariant measure on $(\mathbb{R}^2, \mathcal{B}_2)$ is of the form $c \times \lambda_2$ for some constant c (which was given as a hint) our μ is a constant multiple of λ_2, say $\mu = c\lambda_2$. But for the unit ball centred at the origin (denoted B) we have $\rho(B) = B$, by part (a). Therefore $\mu(B) = \lambda_2(\rho(B)) = \lambda_2(B)$, and hence $c = 1$.

28. (a) Show that $\lambda_2(L) = 0$ for any line segment $L \subset \mathbb{R}^2$.

(b) Let $r > 0$ and set $D := \{x \in \mathbb{R}^2 : |x| < r\}$, the open disk of radius r in $\mathbb{R}^2$ centred on the origin (we define $|x|$ as in the previous question). By approximating to D by an increasing sequence of regular polygons contained in D, show that $\lambda_2(D) = \pi r^2$.

You may use without proof the 'half base times height' formula for the Lebesgue measure (area) of a triangle. You may also use without proof the fact that $(\sin x)/x \to 1$ *as* $x \downarrow 0$.

(a) There exists a rotation ρ (about the origin) such that $\rho(L)$ is a vertical line segment. By rotation invariance (Question 27) $\lambda_2(L) = \lambda_2(\rho(L))$, so without loss of generality we may assume that L is vertical, i.e. $L \subset \{a\} \times (-b, b]$ for some $a, b \in \mathbb{R}$ with $b > 0$.

Given $\varepsilon > 0$, we have $L \subset (a - \varepsilon, a] \times (-b, b]$. Therefore by definition 8.10,

$$\lambda_2(L) \leq \lambda_2((a - \varepsilon, a] \times (-b, b] = \varepsilon(2b).$$

Since ε is arbitrarily small, therefore $\lambda_2(L) = 0$.

(b) For $n \in \mathbb{N}$, let $k_n = 3 \times 2^{n-1}$, so $k_1 = 3$ and $k_{n+1} = 2k_n$. Let P_n be an open regular polygon with k_n sides and vertices equally spaced on the boundary of D. Also assume P_n is oriented in such a way that the vertices of P_{n+1} include all the vertices of P_n for each n. Then $P_n \subset P_{n+1}$ for each n and $D = \cup_{n=1}^{\infty} P_n$ so by upward continuity $\lambda_2(D) = \lim_{n\to\infty} \lambda_2(P_n)$.

We can divide P_n into k_n isosceles triangles, each of which has one vertex at the origin and the other two given by two adjacent vertices of P_n. By rotation invariance, these all have the same area (i.e., 2-dimensional Lebesgue measure). Let T_n be one of these isosceles triangles. Using additivity and also the fact that line segments have zero area by part (a), we have $\lambda_2(P_n) = k_n\lambda_2(T_n)$.

Let $\alpha_n = 2\pi/k_n$, which is the angle between the two 'long edges' of length r of the isosceles triangle T_n. Without loss of generality we may assume one of these two long edges of T_n is the base of T_n. Then the height of T_n is $r \sin \alpha_n$. Therefore by the 'half base times height' formula,

$$\lambda_2(T_n) = (1/2)r \times r \sin \alpha_n$$

and therefore by the second hint,

$$\lambda_2(D) = \lim_{n\to\infty} (k_n/2)r^2 \sin(2\pi/k_n) = \pi r^2 \lim_{n\to\infty} ((k_n/(2\pi)) \sin(2\pi/k_n)) = \pi r^2.$$

29. Suppose F is a function with the properties assumed in Exercise 25.

(a) Prove that there is a unique measure μ_F on $(\mathbb{R}, \mathcal{B})$ with the property that $\mu_F((a, b]) = F(b) - F(a)$ for all $a, b \in \mathbb{R}$ with $a < b$. (You may assume without proof Carathéodory's extension theorem, along with the results of Exercise 25).

As in Exercise 25, for $(a, b] \in \mathcal{I}$ set $\lambda_F((a, b]) = F(b) - F(a)$. Also set $\lambda_F(\varnothing) = 0$.

To be able to check the conditions of the Extension theorem, we need to check that $\pi := \lambda_F$ is a pre-measure on $\mathcal{I}$. Clearly $\pi(\varnothing) = \lambda_F(\varnothing) = 0$, and it remains to check finite additivity and countable subadditivity of π.

For finite additivity let $I, I_1, \ldots, I_k \in \mathcal{I}$ with $I = \cup_{i=1}^k I_i$ and $I_1, \ldots, I_k$ pairwise disjoint. Then by Exercise 25(c), we have $\lambda_F(I) = \sum_{i=1}^k \lambda_F(I_i)$. This verifies the finite additivity.

We also need to check countable sub-additivity of π on $\mathcal{I}$. Let $I, I_1, I_2, \ldots \in \mathcal{I}$ with $I \subset \cup_{k=1}^\infty I_k$. Then by Exercise 25(d), $\lambda_F(I) \leq \sum_{i=1}^\infty \lambda_F(I_i)$. which is the countable subdaditivty of $\pi := \lambda_F$ on $\mathcal{I}$.

Therefore π is a pre-measure on the semiring $\mathcal{I}$, so by the Caratheodory extension theorem it extends to a measure μ_F on $\sigma(\mathcal{I}) = \mathcal{B}$. Also μ_F is σ-finite on $\mathcal{I}$ since $\mu_F((-n, n]) < \infty$ for all n, so by the Uniqueness lemma there is no other measure on $\sigma(\mathcal{I}) = \mathcal{B}$ agreeing with μ_F on $\mathcal{I}$.

(b) Given $y \in \mathbb{R}$, show that the μ_F-measure of the one-point set $\{y\}$ is $\mu_F(\{y\}) = F(y) - F(y-)$, where $F(y-) = \lim_{z\uparrow y} F(z)$.

Set $A_n := (y - 1/n, y]$. Then $A_n \supset A_{n+1}$ for all n, and $A_n \in \mathcal{B}$ for all n, and $\mu_F(A_1) = F(y) - F(y - 1) < \infty$. Also $\{y\} = \cap_{n=1}^\infty A_n$. Therefore by the downward continuity of the measure μ_F, we have

$$\mu(\{y\}) = \mu(\cap_{n=1}^\infty A_n) = \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} (F(y) - F(y - 1/n)) = F(y) - F(y-).$$

(c) Show that $\mu_F([a, b]) = F(b) - F(a-)$, and also find the formulas for $\mu_F((a, b))$ and $\mu_F([a, b))$, when $-\infty < a < b < \infty$.

Since $[a, b] = \{a\} \cup (a, b]$, a disjoint union, using the previous part we have

$$\mu([a, b]) = \mu(\{a\}) + \mu((a, b]) = (F(a) - F(a-)) + (F(b) - F(a)) = F(b) - F(a-).$$

Also

$$\mu((a, b)) = \mu((a, b]) - \mu(\{b\}) = F(b) - F(a) - (F(b) - F(b-)) = F(b-) - F(a).$$

Finally $\mu([a, b)) = \mu((a, b)) + \mu(\{a\}) = F(b-) - F(a-)$.

30. Prove that if $W \subset \mathbb{R}$ is a Borel set, and $f : W \to \mathbb{R}$ is a nondecreasing function, then f is Borel-measurable.

    We need to show for all $\alpha \in \mathbb{R}$ that $f^{-1}((\alpha, \infty]) \in \mathcal{B}$.

    Fix $\alpha \in \mathbb{R}$. Let $T := f^{-1}((\alpha, \infty]) = \{x \in \mathbb{R} : f(x) > \alpha\}$. If $T = \varnothing$ then $T \in \mathcal{B}$. So assume T is non-empty and let $t = \inf(T)$ (or $t = -\infty$ if T is not bounded below).

    If $x \in W$ with $x > t$, then there exists $y < x$ with $y \in T$, so $f(y) > \alpha$, so $f(x) > \alpha$ since f is increasing, so $x \in T$.

    If $x < t$, then $x \notin T$ since t is a lower bound for T.

    Therefore T is either the set $(t, \infty) \cap W$ or the set $[t, \infty) \cap W$. But both (t, ∞) and $[t, \infty)$ are in $\mathcal{B}$, and also we assume $W \in \mathcal{B}$, so $(t, \infty) \cap W \in \mathcal{B}$ and $[t, \infty) \cap W \in \mathcal{B}$. Thus $T \in \mathcal{B}$.