21. Show that λ_1 has the scaling property: for any real number $c \neq 0$ and any Borel set $B \in \mathcal{B}$, we have $\lambda_1(cB) = |c|\lambda_1(B)$. [Here $cB := \{cx : x \in B\}$.]

Define $\nu(B) := \lambda_1(cB)$, $B \in \mathcal{B}$. It is easy to check that ν is a measure; indeed, $\nu(\varnothing) = \lambda_1(c\varnothing) = \lambda_1(\varnothing) = 0$, and if $A_1, A_2, \ldots \in \mathcal{B}$ are pairwise disjoint then so are $cA_1, cA_2, \ldots$ and hence

$$\nu(\cup_{n=1}^{\infty} A_n) = \lambda_1(c \cup_{n=1}^{\infty} A_n) = \lambda_1(\cup_{n=1}^{\infty}(cA_n)) = \sum_{n=1}^{\infty} \lambda_1(cA_n) = \sum_{n=1}^{\infty} \nu(A_n).$$

Likewise, $|c|\lambda_1(B)$ is a measure by Exercise 9 (c). We check that these two measures agree on the class $\mathcal{I}$ of bounded half-open intervals: if $I \in \mathcal{I}$, writing $I = (a, b]$, we have

$$cI = (ca, cb] \quad \text{if } c > 0; \qquad cI = [cb, ca) \quad \text{if } c < 0.$$

In either case the length is $|c|(b-a)$, so $\nu(I) = \lambda_1(cI) = |c|(b-a) = |c|\lambda_1(I)$. Also $\mathcal{I}$ is a π-system, and $\mathbb{R} = \cup_{n=1}^{\infty}(-n, n]$ (a countable union of sets in $\mathcal{I}$ with finite λ_1-measure). Hence the Uniqueness Lemma (Theorem 5.5) implies that $\nu(A) = |c|\lambda_1(A)$ for all Borel sets A.

ALTERNATVE METHOD using outer measure. Since B is Borel we have $\lambda_1(B) = \lambda^*(B)$, the Lebesgue outer measure of B. Assume for now that $\lambda_1(B) < \infty$. Then by definition of outer measure, given $\varepsilon > 0$ we can find intervals $I_1, I_2 \ldots \in \mathcal{I}$ such that $B \subset \cup_{i=1}^{\infty} I_i$ and $\sum_{i=1}^{\infty} \lambda(I_i) < \lambda_1(B) + \varepsilon$. But then also $cB \subset \cup_{i=1}^{\infty} cI_i$ and it is easy to see that $\lambda_1(cI) = |c|\lambda(I)$ for all $I \in \mathcal{I}$ (using Question 14 when $c < 0$). Therefore by countable subadditivity of measure (Theorem 3.3(iii)),

$$\lambda_1(cB) \leq \sum_{i=1}^{\infty} \lambda_1(cI_i) = \sum_{i=1}^{\infty} |c|\lambda(I_i) \leq |c|(\lambda_1(B) + \varepsilon).$$

Therefore since $\varepsilon > 0$ is arbitrary we have $\lambda_1(cB) \leq |c|\lambda_1(B)$, and this is also true when $\lambda_1(B) = \infty$. But applying the same argument using the set cB and the constant c^{-1}, we have that $\lambda_1(B) = \lambda_1(c^{-1}(cB)) \leq |c^{-1}|\lambda_1(cB)$, and hence $\lambda_1(cB) \geq |c|\lambda_1(B)$. Combining the inequalities shows that $\lambda_1(cB) = |c|\lambda_1(B)$.

22. Suppose μ is a translation invariant measure on $(\mathbb{R}, \mathcal{B})$. Set $\gamma := \mu((0, 1])$ and assume $0 < \gamma < \infty$.

(a) Show that $\mu((0, 1/n]) = \gamma/n$ for all $n \in \mathbb{N}$.
Since $(0, 1] = \cup_{i=1}^{n}(\frac{i-1}{n}, \frac{i}{n}]$ (disjoint union) and μ is a measure we have

$$\gamma = \mu((0, 1]) = \sum_{i=1}^{n} \mu((\frac{i-1}{n}, \frac{i}{n}]) = n\mu((0, \frac{1}{n}])$$

where we have used tranlation invariance in the last step. This gives the result.

(b) Show that $\mu((0, q]) = \gamma q$ for all rational $q > 0$.
For any such q we can write $q = m/n$ with $m \in \mathbb{N}, n \in \mathbb{N}$. Then $(0, q] = \cup_{i=1}^{m}(\frac{i-1}{n}, \frac{i}{n}]$ (disjoint union) so that using translation invariance, and (a), we have

$$\mu((0, q]) = \sum_{i=1}^{m} \mu((\frac{i-1}{n}, \frac{i}{n}]) = m\mu((0, \frac{1}{n}]) = m(\gamma/n) = \gamma q.$$

(c) Let $\mathcal{I}'$ be the class of half-open intervals in $\mathbb{R}$ with rational endpoints, i.e. the class of intervals of the form $(q, r]$ with $q \in \mathbb{Q}$, $r \in \mathbb{Q}$ and $q \leq r$. Show that $\mu(I) = \gamma\lambda_1(I)$ for all $I \in \mathcal{I}'$.

For $I = (q, r]$ with $q < r$ and $q, r \in \mathbb{Q}$, by translation invariance $\mu(I) = \mu((0, r - q])$. Since $r - q$ is also rational, we therefore have by part (b) that $\mu(I) = \gamma(r - q) = \gamma\lambda_1(I)$. Obviously if $I = \varnothing$ then $\mu(I) = \gamma\lambda_1(I) = 0$.

(d) Show that $\sigma(\mathcal{I}') = \mathcal{B}$. You may use without proof the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, that is, every non-empty open interval in $\mathbb{R}$ contains at least one rational number.

Let $\mathcal{I}$ be the class of bounded half-open intervals as defined in lectures. By a result from lectures $\mathcal{B} = \sigma(\mathcal{I})$. In particular $\mathcal{I} \subset \mathcal{B}$. Therefore $\mathcal{I}' \subset \mathcal{I} \subset \mathcal{B}$ and hence (since $\mathcal{B}$ is a σ-algebra) $\sigma(\mathcal{I}') \subset \mathcal{B}$.

Conversely, given $I = (a, b] \in \mathcal{I}$, for all $x \in (a, \infty)$ we can find rational q, r with $a < q < x < r$, and hence

$$(a, \infty) = \cup_{\{q,r \in \mathbb{Q}: a < q < r < \infty\}} (q, r]$$

which is a countable union of sets in $\mathcal{I}'$, and therefore is in $\sigma(\mathcal{I}')$. Similarly $(b, \infty) \in \sigma(\mathcal{I}')$, so $I = (a, \infty) \setminus (b, \infty) \in \sigma(\mathcal{I}')$. Hence $\mathcal{I} \subset \sigma(\mathcal{I}')$. Therefore since $\sigma(\mathcal{I}')$ is a σ-algebra, $\mathcal{B} = \sigma(\mathcal{I}) \subset \sigma(\mathcal{I}')$. Combined with the previous paragraph this shows $\sigma(\mathcal{I}') = \mathcal{B}$

(e) Use the Uniqueness lemma to show that $\mu(B) = \gamma\lambda_1(B)$ for all $B \in \mathcal{B}$.

Since λ_1 is a measure on $(\mathbb{R}, \mathcal{B})$, also $\gamma\lambda_1$ is a measure on $(\mathbb{R}, \mathcal{B})$ (see Exercise 9(c)). By part (c), the measures μ and $\gamma\lambda_1$ agree on $\mathcal{I}'$ which is a π-system (because if q, r, s, t are all rational with $q \leq r$ and $s \leq t$ then $(q, r\rangle \cap (s, t\rangle = (\max(q, s), \min(r, t)\rangle \in \mathcal{I}'$).

Moreover, setting $F_n = (-n, n]$ we have $F_n \in \mathcal{I}'$ and $\gamma\lambda_1(F_n) = 2\gamma n < \infty$ for all n, and $\cup_{n=1}^{\infty} F_n = \mathbb{R}$, so $\gamma\lambda_1$ is σ-finite on $\mathcal{I}'$.

Therefore we can apply the Uniqueness lemma to deduce that the measures $\gamma\lambda_1$ and μ agree on $\sigma(\mathcal{I}') = \mathcal{B}$ (using part (d)), that is $\gamma\lambda_1(B) = \mu(B)$ for all $B \in \mathcal{B}$.

23. Suppose X is a non-empty set and $\mathcal{S}$ is a semi-algebra in X. As in Chapter 6 of the notes, let $\mathcal{U}$ be the class of sets of the form $\cup_{i=1}^{k} A_i$ with $k \in \mathbb{N}$ and $A_1, \ldots, A_k$ pairwise disjoint sets in $\mathcal{S}$.

(a) Show by induction on k that if $A \in \mathcal{U}$ then $A^c \in \mathcal{U}$, i.e. $\mathcal{U}$ is closed under complementation.

(b) Show also that $\mathcal{U}$ is closed under pairwise intersections and deduce that $\mathcal{U}$ is an algebra.

(c) Deduce that $\mathcal{U}$ is the algebra generated by $\mathcal{S}$.

(a) For the base case with $k = 1$, note that if $A \in \mathcal{S}$ then by the definition of semi-algebra A^c is a finite union of pairwise disjoint sets in $\mathcal{S}$, and hence is in $\mathcal{U}$. For the inductive hypothesis let $n \geq 1$ and assume if $A = \cup_{i=1}^{n} I_i$ with $A_1, \ldots, A_n$ pairwise disjoint in $\mathcal{S}$, then $A^c \in \mathcal{U}$.

For the inductive step let $A = \cup_{i=1}^{n+1} I_i$ with $I_1, \ldots, I_{n+1}$ disjoint sets in $\mathcal{S}$. Set $B = \cup_{i=1}^{n} I_i$. By the inductive hypothesis we have $B^c \in \mathcal{U}$. Moreover by the base case (or the definition of semi-algebra) we have $I_{n+1}^c \in \mathcal{U}$. Therefore by De Morgan's law, and the first part of Part (b) of this question (to be proved below),

$$A^c = (B \cup I_{n+1})^c = B^c \cap I_{n+1}^c \in \mathcal{U}.$$

This completes the induction.

(b) Suppose $A, B \in \mathcal{U}$; we shall show $A \cap B \in \mathcal{U}$. Write $A = \cup_{i=1}^k I_i$ and $B = \cup_{j=1}^\ell J_j$ with $I_1, \ldots, I_k$ pairwise disjoint sets in $\mathcal{S}$, and $J_1, \ldots, J_\ell$ pairwise disjoint sets in $\mathcal{S}$. Then

$$A \cap B = \cup_{i=1}^k \cup_{j=1}^\ell (I_i \cap J_j) = \cup_{(i,j) \in [k] \times [\ell]} K_{i,j}$$

where we set $K_{i,j} := I_i \cap J_j$ for each (i,j). Since $\mathcal{S}$ is a π-system we have $K_{i,j} \in \mathcal{S}$ for each (i,j). Also for $(i', j') \neq (i,j)$ we have

$$K_{i,j} \cap K_{i',j'} = I_i \cap J_j \cap I_{i'} \cap J_{j'} = \varnothing,$$

because either $i \neq i'$ (so $I_i \cap I_{i'} = \varnothing$) or $j \neq j'$ (so $J_j \cap J_{j'} = \varnothing$).

Therefore the sets $K_{i,j}, 1 \leq i \leq k, 1 \leq j \leq \ell$ are pairwise disjoint so $A \cap B \in \mathcal{U}$. Therefore $\mathcal{U}$ is closed under pairwise intersections (that is, it is a π-system). This gives the first part of (b), and hence also (a).

Next we show $\mathcal{U}$ is closed under pairwise unions. Given $A, B \in \mathcal{U}$ we have (by part (a)) that $A^c \in \mathcal{U}$ and $B^c \in \mathcal{U}$, so that by the De Morgan law, and the first part of (b),

$$(A \cup B)^c = A^c \cap B^c \in \mathcal{U}$$

and hence (by (a) again) also $A \cup B = ((A \cup B)^c)^c \in \mathcal{U}$. So $\mathcal{U}$ is an algebra.

(c) Let $\mathcal{A}$ be the algebra generated by $\mathcal{S}$. Then $\mathcal{A}$ is the intersection of all algebras containing $\mathcal{S}$ so $\mathcal{A} \subset \mathcal{U}$ (since we've just shown that $\mathcal{U}$ is one such algebra).

Conversely, since $\mathcal{A}$ is an algebra we have for any $I_1, \ldots, I_k \in \mathcal{S}$ that $\cup_{i=1}^k I_i \in \mathcal{A}$. Therefore $\mathcal{U} \subset \mathcal{A}$ so $\mathcal{U} = \mathcal{A}$.

24. Suppose X is a non-empty set, $\mathcal{S}$ is a semi-algebra in X and π is a pre-measure on $(X, \mathcal{S})$.

   (a) Show that if $A, A_1, \ldots, A_k \in \mathcal{S}$ with $A_1, \ldots, A_k$ pairwise disjoint and $\cup_{i=1}^k A_i \subset A$, then $\sum_{i=1}^k \pi(A_i) \leq \pi(A)$.
   Define $\mathcal{U}$ as in Question 23. By that question $\mathcal{U}$ is an algebra. Then $\cup_{i=1}^k A_i \in \mathcal{U}$, so $(\cup_{i=1}^k A_i)^c \in \mathcal{U}$, so $B := A \cap (\cup_{i=1}^k A_i)^c \in \mathcal{U}$. Writing $B = \cup_{j=1}^m C_j$ with $C_1, \ldots, C_m$ pairwise disjoint sets in $\mathcal{S}$, we have $A = (\cup_{i=1}^k A_i) \cup (\cup_{j=1}^m C_j)$ which is a finite union of disjoint sets in $\mathcal{S}$, so $\pi(A) = (\sum_{i=1}^k \pi(A_i)) + \sum_{j=1}^m \pi(C_j)$ since π is a pre-measure so finitely additive. Hence $\pi(A) \geq \sum_{i=1}^k \pi(A_i)$.

   (b) Show that π is countably additive, i.e. $\pi(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \pi(A_n)$ whenever $A_1, A_2, \ldots \in \mathcal{S}$ are pairwise disjoint with $\cup_{n=1}^\infty A_n \in \mathcal{S}$.
   Set $A = \cup_{n=1}^\infty A_n$. By (a), $\sum_{n=1}^k \pi(A_n) \leq \pi(A)$ for all k, so taking the large-k limit, $\sum_{n=1}^\infty \pi(A_n) \leq \pi(A)$. Conversely, since π is a pre-measure so countably sub-additive, $\pi(A) \leq \sum_{n=1}^\infty \pi(A_n)$. The result follows.

25. Let $F : (-\infty, \infty) \to \mathbb{R}$ be a non-decreasing, right continuous function. Let $\mathcal{I}$ denote the set of bounded half-open intervals in $\mathbb{R}$ (as in lectures). Put $\lambda_F(\varnothing) = 0$, and for all other $I \in \mathcal{I}$, put

$$\lambda_F(I) = F(b) - F(a), \quad \text{where } I \text{ has endpoints } a \text{ and } b.$$

(a) Check that $\lambda_F(I) \geq 0$ for all $I \in \mathcal{I}$.
(b) Show that λ_F is finitely sub-additive on $\mathcal{I}$.
(c) Show that λ_F is finitely additive on $\mathcal{I}$.
(d) Show that λ_F is countably additive on $\mathcal{I}$.

(a) If $I = \varnothing$ then $\lambda_F(I) = 0$ (this should have been stated more clearly in the question).

If $I = (a, b]$ for finite $a < b$, then $\lambda_F(I) = F(b) - F(a) \geq 0$ since F is non-decreasing.

(b) We adapt the proof of Lemma 4.3(i). We need to show that if $A \subset \cup_{i=1}^k A_i$ with all of $A, A_1, \dots, A_k$ in $\mathcal{I}$, then $\lambda_F(A) \leq \sum_{i=1}^k \lambda_F(A_i)$. It is enough to prove this in the case where $A, A_1, \dots, A_k$ are all non-empty, so that we can and do write $A = (a, b]$ and $A_i = (a_i, b_i]$ for $1 \leq i \leq k$. We may also assume without loss of generality that $b_k \geq \max_{1 \leq i \leq k-1} b_i$.

We prove the result by induction on k. If $A \subset A_1$ then $a \geq a_1$ and $b \leq b_1$, so $F(a) \geq F(a_1)$ and $F(b) \leq F(b_1)$ (since F is assumed non-decreasing), and hence $\lambda_F(A_1) - \lambda_F(A) = F(b_1) - F(a_1) - F(b) + F(a) \geq 0$, so the statement holds if $k = 1$.

Suppose now that $k \geq 2$ and that the statement holds for $k - 1$. Suppose $A \subset \cup_{i=1}^k A_i$. If $A \cap A_k = \varnothing$ then $A \subset \cup_{i=1}^{k-1} A_i$ so by the inductive hypothesis $\lambda_F(A) \leq \sum_{i=1}^{k-1} \lambda_F(A_i) \leq \sum_{i=1}^k \lambda_F(A_i)$, completing the induction in this case.

If $A \cap A_k \neq \varnothing$ then $a_k < b \leq b_k$ and $(a, a_k] \subset \cup_{i=1}^{k-1} A_i$ (where we define $(a, a_k] = \varnothing$ if $a_k \leq a$). Hence by the inductive hypothesis and the fact that F is non-decreasing,

$$\lambda_F(A) = (F(b) - F(a_k)) + (F(a_k) - F(a)) \leq \lambda_F(A_k) + \sum_{i=1}^{k-1} \lambda_F(A_i) = \sum_{i=1}^{k} \lambda_F(A_i),$$

completing the induction.

(c) We need to show that for all $k \in \mathbb{N}$, if $I = \cup_{i=1}^k I_i$ with $I, I_1, \dots, I_k$ all in $\mathcal{I}$ and $I_1, \dots, I_k$ pairwise disjoint, then $\lambda_F(I) = \sum_{i=1}^k \lambda_F(I_i)$.

As in part (b), it suffices to prove this for the case where $I, I_1, \dots, I_k$ are all non-empty. We can then write $I = (a, b]$ and $I_i = (a_i, b_i]$ for each i, and assume without loss of generality that $b_1 \leq b_2 \leq \cdots \leq b_k$.

Since $I \subset \cup_{i=1}^k I_i$ we have $a_1 \leq a$ and $b_k \geq b$. Since $I_1 \subset I$ and $I_k \subset I$ we have $a_1 \geq a$ and $b_k \leq b$. Thus $a_1 = a$ and $b_k = b$.

For $1 \leq i \leq k - 1$, we have $b_i \leq a_{i+1}$ (because $I_i \cap I_{i+1} = \varnothing$) and $b_i \geq a_{i+1}$ (because otherwise, the interval (b_i, a_{i+1}) would be non-empty and contained in $I \setminus \cup_{j=1}^k I_j$, contradicting the assumption $I = \cup_{j=1}^k I_j$). Therefore $b_i = a_{i+1}$. Hence

$$\begin{aligned}\sum_{i=1}^k \lambda_F(I_i) = \sum_{i=1}^k (F(b_i) - F(a_i)) &= F(b_k) + \left(\sum_{i=1}^{k-1} (F(b_i) - F(a_{i+1}) \right) - F(a_1) \\ &= F(b) + 0 - F(a) = \lambda_F(I),\end{aligned}$$

as required. Alternatively the result can be proved by induction, similarly to part (b).

(d) We modify the proof of Theorem 4.4. Suppose $A, A_1, A_2, \ldots \in \mathcal{I}$ with $A \subset \cup_{i=1}^{\infty} A_i$. We may assume $A \neq \varnothing$ (otherwise the subadditivity result is trivial). Write $A = (a, b]$ and $A_i = (a_i, b_i]$ for each i (we may omit all i with $A_i = \varnothing$ from the list of A_i).

Let $\varepsilon > 0$ be fixed. Since F is right-continuous, we can find $a' \in (a, b)$ such that $F(a') < F(a) + \varepsilon$, and hence with $A' := (a', b]$ we have

$$\lambda_F(A) = F(b) - F(a) \leq F(b) - F(a') + \varepsilon = \lambda_F(A') + \varepsilon. \tag{1}$$

For each $n = 1, 2, \ldots$, by the right continuity of F, we can find $b_n' > b_n$ with $F(b_n') < F(b_n) + \frac{\varepsilon}{2^n}$. Then setting $A_n' := (a_n, b_n']$ we have

$$\lambda_F(A_n') = F(b_n') - F(a_n) \leq F(b_n) + \frac{\varepsilon}{2^n} - F(a) = \lambda_F(A_n) + \frac{\varepsilon}{2^n}. \tag{2}$$

Now

$$[a', b] \subset A \subset \cup_{n=1}^{\infty} A_n \subset \cup_{n=1}^{\infty} (a_n, b_n').$$

Therefore, by compactness of $[a', b]$ (Heine-Borel), there exists $N \geq 1$ such that

$$[a', b] \subset \cup_{n=1}^{N} (a_n, b_n').$$

It follows that $A' = (a', b] \subset \cup_{n=1}^{N} (a_n, b_n']$, so by the finite subadditivity established in part (b),

$$\lambda_F(A') \leq \sum_{n=1}^{N} \lambda_F(A_n').$$

Hence we have

$$\lambda_F(A) \overset{(1)}{\leq} \lambda_F((A') + \varepsilon \leq \left(\sum_{n=1}^{N} \lambda_F(A_n') \right) + \varepsilon \overset{(2)}{\leq} \left(\sum_{n=1}^{N} \lambda_F(A_n) \right) + \varepsilon + \sum_{n=1}^{N} \frac{\varepsilon}{2^n} \leq \left(\sum_{n=1}^{\infty} \lambda_F(A_n) \right) + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we can let $\varepsilon \downarrow 0$, and we obtain the result.