13. Show that if $A \subset \mathbb{R}$ is countable then $A \in \mathcal{B}$ and $\lambda_1(A) = 0$.

    Let $x \in \mathbb{R}$. Then $\mathbb{R} \setminus \{x\} = (-\infty, x) \cup (x, \infty)$ is open, so is in $\mathcal{B}$ (since $\mathcal{B}$ is the σ-algebra generated by the collection $\mathcal{O}$ of open sets in $\mathbb{R}$). Therefore since $\mathcal{B}$ is a σ-algebra, also $\{x\} = (\mathbb{R} \setminus \{x\})^c \in \mathcal{B}$.

    Now suppose $A \subset \mathbb{R}$ is countable. Then we can write $A = \cup_{i=1}^{\infty} \{x_i\}$ for some sequence of real numbers $(x_1, x_2, \ldots)$ (in the case where A is finite we could take $x_i = x_1$ for all but finitely many i). Since $\{x_i\} \in \mathcal{B}$ for each i, and since $\mathcal{B}$ is a σ-algebra, we have

    $$A = \cup_{i=1}^{\infty} \{x_i\} \in \mathcal{B}.$$

    Next we want to show $\lambda_1(A) = 0$. For $x \in \mathbb{R}$, setting $J_n = (x - 1/n, x + 1/n]$ we have $\{x\} \subset J_n$ so $\lambda_1(\{x\}) \leq \lambda_1(J_n) = 2/n$. Since n is arbitrarily large this shows that $\lambda_1(\{x\}) \leq 0$ and hence $\lambda_1(\{x\}) = 0$. Then with A and x_i as above, since $A = \cup_{i=1}^{\infty} \{x_i\}$, using countable subadditivity of Lebesgue measure (see Theorem 3.3) we have

    $$\lambda_1(A) \leq \sum_{i=1}^{\infty} \lambda_1(\{x_i\}) = 0.$$

    Also $\lambda_1(A) \geq 0$ since λ_1 is a measure. So $\lambda_1(A) = 0$.

14. Show that for any interval I with left endpoint a and right endpoint b we have $\lambda_1(I) = b - a$ (regardless of whether $a, b \in I$ or not).

    Assume $a < b$ (in the degenerate case $a = b$, either $I = \{a\}$ so $\lambda_1(I) = 0$ by the previous question, or $I = \varnothing$ so $\lambda_1(I) = 0$).

    Choose $\varepsilon \in (0, (b - a)/2)$. Let $I_0 = (a + \varepsilon, b - \varepsilon]$ and $I_1 = (a - \varepsilon, b + \varepsilon]$.

    Then $I_0 \subset I \subset I_1$ so that $\lambda_1(I_0) \leq \lambda_1(I) \leq \lambda_1(I_1)$.

    λ_1 is defined as the unique measure on $(\mathbb{R}, \mathcal{B})$ such that $\lambda_1((u, v]) = v - u$ for all $u < v$. Therefore

    $$\lambda_1(I_0) = (b - \varepsilon) - (a + \varepsilon) = b - a - 2\varepsilon; \quad \lambda_1(I_1) = (b + \varepsilon) - (a - \varepsilon) = b - a + 2\varepsilon,$$

    and therefore

    $$b - a - 2\varepsilon \leq \lambda_1(I) \leq b - a + 2\varepsilon.$$

    Since ε can be arbitrarily small (subject to $\varepsilon > 0$) this shows that $\lambda_1(I) = b - a$.

15. Give an example of a Borel set $A \subset \mathbb{R}$ with $\lambda_1(A) > 0$ but with no non-empty open interval contained in A.

    We could take $A = (0, 1] \setminus \mathbb{Q}$. Note that $\mathbb{Q} \in \mathcal{B}$ since $\mathbb{Q}$ is countable. Also, the set $\mathbb{Q} \cap (0, 1]$ is countable, so $\lambda_1(\mathbb{Q} \cap (0, 1]) = 0$. Since $A \cup (\mathbb{Q} \cap (0, 1]) = (0, 1]$ (disjoint union), we have $\lambda_1(A) + \lambda_1(\mathbb{Q} \cap (0, 1]) = \lambda_1(0, 1] = 1$, so

    $$\lambda_1(A) = 1 - \lambda_1(\mathbb{Q} \cap (0, 1]) = 1 > 0.$$

    However, A does not contain any non-empty open interval since $\mathbb{Q}$ is dense in the real line, so any non-empty interval of the form (a, b) contains at least one element of A.

16. Given $\varepsilon > 0$, give an example of an open set $U \subset \mathbb{R}$ with $\lambda_1(U) < \varepsilon$ that is dense in $\mathbb{R}$, i.e. has non-empty intersection with every non-empty open interval in $\mathbb{R}$.

    Enumerate the rationals $\mathbb{Q}$ as $\mathbb{Q} = \{q_1, q_2, q_3, \ldots\}$ (this can be done because $\mathbb{Q}$ is countably infinite). Let $\delta \in (0, \varepsilon/2)$. For $i \in \mathbb{N}$ Set

    $$I_i := (q_i - \frac{\delta}{2^i}, q_i + \frac{\delta}{2^i}),$$

    and set $U := \cup_{i=1}^{\infty} I_i$. Then U is open since if $x \in U$, we can find i such that $x \in I_i$, but then since I_i is an open interval we can find $\eta > 0$ with $(x - \eta, x + \eta) \subset I_i \subset U$.

    Also, U is dense in $\mathbb{R}$ because $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{Q} \subset U$.

    Finally, by subadditivity of Lebesgue measure λ_1,

    $$\lambda_1(U) \leq \sum_{i=1}^{\infty} \lambda_1(I_i) = \sum_{i=1}^{\infty} \frac{2\delta}{2^i} = 2\delta,$$

    which is less than ε by our choice of δ.

17. Suppose $A \subset \mathbb{R}$ is a bounded Borel set. Show that for all $\varepsilon > 0$ there exists a set U which is a finite union of intervals, such that $\lambda_1(A \triangle U) < \varepsilon$, where $A \triangle U := (A \setminus U) \cup (U \setminus A)$.

    Since $\lambda_1(A) = \lambda^*(A)$ (the Lebesgue outer measure of A), which is finite because A is bounded, by the definiton of Lebesgue outer measure there exists a sequence of half-open intervals $I_1, I_2, I_3, \ldots$ such that $A \subset \cup_{i=1}^{\infty} I_i := S$, and $\sum_{i=1}^{\infty} \lambda_1(I_i) < \lambda_1(A) + \varepsilon/2$.

    By subadditivity of the measure λ_1 (Theorem 3.3(iii)), $\lambda_1(S) \leq \sum_{i=1}^{\infty} \lambda_1(I_i) < \lambda_1(A) + \varepsilon/2$. In particular $\lambda_1(S) < \infty$.

    For each n set $S_n := \cup_{i=1}^{n} I_i$. Then $S_n \subset S_{n+1}$ for all n, and $\cup_{n=1}^{\infty} S_n = S$. Therefore by the upward continuity of the measure λ_1 (Theorem 3.3(i)) we have $\lambda_1(S_n) \to \lambda_1(S)$ as $n \to \infty$, and since $\lambda_1(S) < \infty$, we can (and do) choose N such that $\lambda_1(S_N) > \lambda_1(S) - \varepsilon/2$.

    Set $U = S_N$. Then $A \subset S$ and $U \subset S$, so that $A \setminus U \subset S \setminus U$, and thus

    $$\lambda_1(A \setminus U) \leq \lambda_1(S \setminus U) = \lambda_1(S) - \lambda_1(U) < \varepsilon/2,$$

    and similarly

    $$\lambda_1(U \setminus A) \leq \lambda_1(S \setminus A) = \lambda_1(S) - \lambda_1(A) < \varepsilon/2.$$

    Combining the last two displays shows that $\lambda_1(U \triangle A) = \lambda_1(U \setminus A) + \lambda_1(A \setminus U) < \varepsilon$.

18. In this question we write $\lambda^*(A)$ for the Lebesgue outer measure of A.

    (a) What is the definition of the *Lebesgue outer measure* of a set $A \subset \mathbb{R}$?

    (b) Show that for any (not necessarily Borel) $A \subset \mathbb{R}$ there exists a Borel set $B \subset \mathbb{R}$ with $A \subset B$ and $\lambda_1(B) = \lambda^*(A)$.

    (c) Suppose $A \subset \mathbb{R}$ is a Borel set with $\lambda_1(A) > 0$. Using the fact that $\lambda_1(A) = \lambda^*(A)$, show that for any $\varepsilon > 0$ there exists a non-empty half-open interval I with $\lambda_1(A \cap I) \geq (1 - \varepsilon)\lambda_1(I)$.

(d) Show that the set $A \ominus A := \{x - y : x, y \in A\}$ includes a non-empty half-open interval.

(a) The Lebesgue outer measure of A is defined by

$$\lambda^*(A) := \inf\{\sum_{i=1}^{\infty}(b_i - a_i) : A \subset \cup_{i=1}^{\infty}(a_i, b_i], -\infty < a_i < b_i < \infty \forall i \in \mathbb{N}\}.$$

(b) By definition of Lebesgue outer measure λ^*, if $\lambda^*(A) < \infty$ then for each $n \in \mathbb{N}$ we can find a covering $I_{n,1}, I_{n,2}, \ldots$ of A by half-open intervals, such that $\sum_{i=1}^{\infty} \lambda(I_{n,i}) \leq \lambda^*(A) + 1/n$.

Let $A_n = \cup_{i=1}^{\infty} I_{n,i}$. Since each interval $I_{n,i}$ is in $\mathcal{B}$, and $\mathcal{B}$ is a σ-algebra we have $A_n = \cup_{i=1}^{\infty} I_{n,i} \in \mathcal{B}$. That is, A_n is a Borel set. By subadditivity of measure,

$$\lambda_1(A_n) \leq \sum_{i=1}^{\infty} \lambda_1(I_i) \leq \lambda^*(A) + n^{-1}.$$

Now set $B = \cap_{n=1}^{\infty} A_n$. Then $B \in \mathcal{B}$ (i.e. B is a Borel set), because it is a countable intersection of sets in $\mathcal{B}$. Then $A \subset B$, but for all n we have $B \subset A_n$ so $\lambda_1(B) \leq \lambda_1(A_n) \leq \lambda^*(A) + 1/n$. Therefore $\lambda_1(B) \leq \lambda^*(A)$, but also $\lambda^*(A) \leq \lambda^*(B) = \lambda_1(B)$ since $A \subset B$, so $\lambda_1(B) = \lambda^*(A)$.

If $\lambda^*(A) = \infty$ then we can just take $B = \mathbb{R}$.

(c) Without loss of generality we may assume A is bounded. For if not, then let $A_n = A \cap (-n, n]$. By upwards continuity $\lambda_1(A_n) \to \lambda_1(A)$ as $n \to \infty$, and since $\lambda_1(A) > 0$, we can choose n with $\lambda_1(A_n) > 0$. If the result holds for this A_n then it also holds for A since $A_n \subset A$. Hence, from now on we assume A is bounded, with $0 < \lambda_1(A) < \infty$.

Proof by contradiction; supose there exists $\varepsilon > 0$ such that for every interval I we have $\lambda_1(I \cap A) < (1 - \varepsilon)\lambda_1(I)$. Now fix this ε. Clearly $\varepsilon < 1$. Let $\delta > 0$ be taken so small that $(1 - \varepsilon)(1 + \delta) < 1$. Then by the defnition of outer measure, we can take a sequence of intervals $I_n \in \mathcal{I}$, defined for each $n \in \mathbb{N}$, such that $A \subset \cup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} \lambda_1(I_n) < (1 + \delta)\lambda_1(A)$. Then

$$\sum_{n=1}^{\infty} \lambda_1(A \cap I_n) \leq (1 - \varepsilon) \sum_{n=1}^{\infty} \lambda_1(I_n) \leq (1 - \varepsilon)(1 + \delta)\lambda_1(A).$$

Since $A \subset \cup_{n=1}^{\infty} I_n$, also $A = A \cap (\cup_{n=1}^{\infty} I_n) = \cup_{n=1}^{\infty}(A \cap I_n)$, so by the countable subadditivity of Lebesgue measure (Theorem 3.3 (iii)),

$$\lambda_1(A) \leq \sum_{n=1}^{\infty} \lambda_1(A \cap I_n).$$

Combining this with the previous displayed inequality shows that $\lambda_1(A) \leq (1 - \varepsilon)(1 + \delta)\lambda_1(A)$, which is a contradiction by the choice of δ.

(d) Here we are assuming A is as in Part (c). Using Part (c) with $\varepsilon = 0.01$, pick $I \in \mathcal{I}$ such that $\lambda_1(A \cap I) \geq 0.99\lambda(I)$. Write $I = (a, a + \delta]$ with $\delta > 0$. Then $\lambda_1(A \cap I) \geq 0.99\delta$.

Suppose $z \notin A \ominus A$. Then for all $x, y \in A$ we have $z \neq x - y$ so $x \neq z + y$ and hence the $(A + z) \cap A = \varnothing$, that is $A + z \subset A^c$.

Suppose moreover that $z \in (0, \delta/2]$. Then $(a, a+\delta/2] + z \subset (a, a+\delta]$ since if $w \in (a, a+\delta/2]$ then $a + 0 < w + z \leq (a + \delta/2) + \delta/2$.

Therefore $(A \cap (a, a+\delta/2]) + z \subset A^c \cap (a, a+\delta]$. Thus by the translation invariance of Lebesgue measure,

$$\begin{aligned} \lambda_1(A^c \cap (a, a+\delta]) \geq \lambda_1((A \cap (a, a+\delta/2]) + z) &= \lambda_1(A \cap (a, a+\delta/2]) \\ &= (\delta/2) - \lambda_1(A^c \cap (a, a+\delta/2]) \\ &\geq 0.49\delta. \end{aligned}$$

On the other hand $\lambda_1(A^c \cap (a, a+\delta]) = \delta - \lambda_1(A \cap (a, a+\delta]) \leq 0.01\delta$, and these two inequalities for $\lambda_1(A^c \cap (a, a+\delta])$ are contradictary.

Therefore no such z exists, i.e. no z satisfies both $z \notin A \ominus A$ and $z \in (0, \delta/2]$, or in other words $(0, \delta/2] \subset A \ominus A$.

19. Suppose X is a non-empty set and $\mathcal{D}$ is a π-system in X. Show that for any $k \in \mathbb{N}$, if $A_i \in \mathcal{D}$ for $i = 1, 2, \ldots, k$ then $\cap_{i=1}^k A_i \in \mathcal{D}$.

    Proof by induction. The result is true for $k = 1$. Suppose it is true for some k. Suppose $A_i \in \mathcal{D}$ for $1 \leq i \leq k+1$. Set $B = \cup_{i=1}^k A_i$. By the inductive hypothesis $B \in \mathcal{D}$. Since $\mathcal{D}$ is a π-system

    $$\cap_{i=1}^{k+1} A_i = B \cap A_{i+1} \in \mathcal{D}$$

    which completes the induction.

20. Let $\mathcal{I}$ denote the class of half-open intervals in $\mathbb{R}$, together with the empty set (as in the lecture notes). Define the set-function: $\pi : \mathcal{I} \to [0, \infty]$ by

    $$\pi(A) := \begin{cases} 0 & \text{if } A = \varnothing; \\ \infty & A \neq \varnothing. \end{cases}$$

    Show that π has *more than one* extension to a measure on $\mathcal{B} = \sigma(\mathcal{I})$. What condition (of the uniqueness theorem) failed here?

    One such extension would be the counting measure μ defined in Example 3.2(a). Since any non-empty interval has infinitely many elements, this μ is also an extension of π to $\mathcal{B}$.

    However, for any constant $c \in (0, \infty]$ the set function $c\mu$ is also a measure, which also extends π.

    For any sequence of sets $F_n \in \mathcal{I}$ with we must have $F_n \neq \varnothing$ for at least one n, and for that n we have $\pi(F_n) = \infty$ (by the definition of π). Hence there is no sequence of sets $F_n \in \mathcal{I}$ with $\cup_{n=1}^\infty F_n = \mathbb{R}$ and $\pi(F_n) < \infty$ for all n. Therefore Theorem 5.5 (Uniqueness lemma) is not applicable.