55. Let $p \in [1, \infty)$ and let $f \in L^p(\mathbb{R})$. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be real-valued sequences such that $\sum_{n=1}^{\infty} |a_n| < \infty$. Show that the sequence of functions

$$f_n(x) := \sum_{k=1}^{n} a_k f(x - b_k)$$

converges in $L^p(\mathbb{R})$.

We show that the sequence is a Cauchy sequence in L^p. For each k set $g_k(x) = f(x - b_k)$ for $x \in \mathbb{R}$. Then $g_k \in L^p$ with $\|g_k\|_p = \|f\|_p$ by Question 37. By Minkowski's inequality and a straightforward induction argument, we have for any $h_1, \dots, h_j \in L^p$ that $\|h_1 + h_2 + \cdots h_j\|_p \leq \sum_{i=1}^{j} \|h_i\|_p$. Hence for any $n < m$ we have

$$\|f_m - f_n\|_p = \left\| \sum_{k=n+1}^{m} a_k g_k \right\|_p \leq \sum_{k=n+1}^{m} \|a_k g_k\|_p$$

and since $\|\alpha h\|_p = |\alpha| \|h\|_p$ for any real α and any $h \in L^p$, we therefore have for $n < m$ that

$$\|f_m - f_n\|_p \leq \sum_{k=n+1}^{m} |a_k| \|g_k\|_p = \|f\|_p \sum_{k=n+1}^{m} |a_k| \leq \|f\|_p \sum_{k=n+1}^{\infty} |a_k|$$

which tends to zero as $n \to \infty$ since it is the tail of a convergent series. This shows that f_n is a Cauchy sequence in L^p, so by the Riesz-Fischer theorem f_n converges in L^p to a limit function in L^p.

56. Suppose $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are sequences of nonnegative numbers, such that $A := \sum_{n=1}^{\infty} a_n^{4/3} < \infty$ and $B := \sum_{n=1}^{\infty} b_n^4 < \infty$. Show that $\sum_{n=1}^{\infty} a_n b_n \leq A^{3/4} B^{1/4}$. (you may use results from lectures without proof).

We apply Hölder's inequality on the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, where μ is the counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, with $p = 4/3$ and $q = 4$ so $1/p + 1/q = 1$. Setting $f(n) = a_n$ and $g(n) = b_n$ for $n \in \mathbb{N}$, using Question 38 we have $\int fg d\mu = \sum_n a_n b_n$, and $\int f^p d\mu = A$ and $\int g^q d\mu = B$. Hence by Hölder's inequality

$$\sum_{n=1}^{\infty} a_n b_n = \int fg d\mu \leq \left(\int f^p d\mu \right)^{1/p} \left(\int g^q d\mu \right)^{1/q} = A^{1/p} B^{1/q}.$$

57. Suppose that $(X, \mathcal{M}, \mu)$ is a measure space, and $1 \leq p < q < \infty$.
    (a) Show that if μ is a probability measure and $f \in L^q(\mu)$, then $\|f\|_p \leq \|f\|_q$.
    [Hint: note that $f = f \cdot 1$, and apply Hölder's inequality]
    (b) Show that if $\mu(X) < \infty$ then $L^q(\mu) \subset L^p(\mu)$.
    (c) Give an example to show that if $\mu(X) = \infty$, then we might not have $L^q(\mu) \subset L^p(\mu)$.

(a) Following the hint, set $g(x) = 1$ for all $x \in X$. Assume $f \in L^q(\mu)$. Then $f^p = f^p \cdot g$, so by Hölder's inequality

$$\|f\|_p^p = \int |f|^p d\mu = \|f^p \cdot g\|_1 \leq \|(f^p)\|_{q/p} \|g\|_r$$

where r is the conjugate exponent to q/p. Since $g \equiv 1$ and μ is a probability measure, $\|g\|_r = (\int 1 d\mu)^{1/r} = 1$, so

$$\|f\|_p^p \leq \left(\int (|f|^p)^{q/p} \right)^{p/q} = \left(\int |f|^q \right)^{p/q} = \|f\|_q^p$$

so the result follows.

(b) If $\mu(X)$ is finite (but now not equal to 1), we can still take $g \equiv 1$ as above; with r as above we now have that $\|g\|_r = \mu(X)^{1/r} < \infty$. Therefore repeating the argument above shows that if $f \in L^q(\mu)$ then

$$\|f\|_p^p \leq \|f\|_q^p \times \|g\|_r < \infty$$

so that $f \in L^p(\mu)$ as required.

(c) We can use part (b) of Question 54. Take $X = [1, \infty)$ with Lebesgue measure and $f(x) = x^{-\beta}$ with $q^{-1} < \beta \leq p^{-1}$, so that $p\beta \leq 1 < q\beta$. Then $f \in L^q \setminus L^p$, so here we do not have $L^q \subset L^p$.

58. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Let $p \in (1, \infty)$. Suppose $f : X \to \mathbb{R}$ and (for all $n \in \mathbb{N}$) $f_n : X \to \mathbb{R}$ are measurable functions, and assume $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$. For all $n \in \mathbb{N}$ and $x \in X$, set

$$g_n(x) = \sum_{k=1}^{n} |f_k(x)| \quad \text{and} \quad g_\infty(x) = \sum_{k=1}^{\infty} |f_k(x)|.$$

(i) Show that $\|g_n\|_p \to \|g_\infty\|_p$ as $n \to \infty$, and deduce that $\|g_\infty\|_p < \infty$.

(ii) Show that the function $h(x) := \sum_{n=1}^{\infty} f_n(x)$ is well-defined and finite μ-a.e., that is, the sum converges for μ-a.e. $x \in X$.

(i) By using Minkowski's inequality repeatedly we have

$$\|g_n\|_p \leq \sum_{k=1}^{n} \||f_k|\|_p = \sum_{k=1}^{n} \|f_k\|_p \leq \sum_{k=1}^{\infty} \|f_k\|_p,$$

which is finite by assumption. Also for each $x \in X$ we have that $(\sum_{k=1}^{n} |f_k(x)|)^p$ is nonnegative and nondecreasing in k. Therefore by definition and by MON,

$$\|g_n\|_p^p = \int_X \left(\sum_{k=1}^{n} |f_k| \right)^p d\mu \to \int_X \left(\sum_{k=1}^{\infty} |f_k| \right)^p d\mu, \qquad \text{as } n \to \infty.$$

Therefore this limit is finite (being the limit of a bounded sequence). The limit in the display above is equal to $\|g_\infty\|_p^p$. Taking pth roots gives us the required result that $\|g_n\|_p \to \|g_\infty\|_p$ and $\|g_\infty\|_p < \infty$.

(ii) Clearly $g_\infty(x) \geq 0$ for all $x \in X$, and the previous part we have $\int (g_\infty(x))^p \mu(dx) < \infty$, so by Lemma 10.5(b), $g(x) < \infty$ for μ-almost all $x \in X$.

Therefore for μ-almost all $x \in X$, we have $\sum_{n=1}^\infty |f_n(x)|$ is convergent so that the sum $\sum_{n=1}^\infty f_n(x)$ converges (absolutely).

[In case you need reminding about absolute convergence: Fix x such that $\sum_{n=1}^\infty |f_n(x)| < \infty$. Given $\varepsilon > 0$ we can choose N so that $\sum_{n=N}^\infty |f_n(x)| < \varepsilon$. Taking $s_n := \sum_{k=1}^n f_k(x)$, for $m > n \geq N$ we have $|s_m - s_n| = |\sum_{k=n+1}^m f_k(x)| \leq \sum_{k=n+1}^m |f_k(x)| < \varepsilon$. Hence $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence so converges to a finite limit, i.e. the series $\sum_{n=1}^\infty f_n(x)$ is convergent.]

59. Let $W \in \mathcal{B}$, and for $f, g \in L^2(W)$, write $\langle f, g\rangle = \int_W f(x)g(x)dx$. Show that if also $h \in L^2(W)$ and $a, b \in \mathbb{R}$ then $\langle f, ag + bh\rangle = a\langle f, g\rangle + b\langle f, h\rangle$.

By linearity of the integral (Theorem 11.5), we have

$$\langle f, ag + bh\rangle = \int_W f(x)(ag(x) + bh(x))dx$$

$$= a \int_W f(x)g(x)dx + b \int_W f(x)h(x)dx = a\langle f, g\rangle + b\langle f, h\rangle.$$

Theorem 11.5 is applicable here because by Hölder's inequality (Theorem 14.9) the function fg satisfies $\|fg\|_1 \leq \|f\|_2\|g\|_2 < \infty$, so $fg \in L^1(W)$ and likewise $fh \in L^1(W)$.

60. For $n \in \mathbb{N}$, let $f_n(x) = \sin(nx)$.

(a) Show that for $n, m \in \mathbb{N}$ with $n \neq m$ we have $\int_0^{2\pi} f_n(x)f_m(x)dx = 0$, while $\int_0^{2\pi}(f_n(x))^2 dx = \pi$. *[Hint: recall that* $\cos(a + b) = \cos a \cos b - \sin a \sin b$*]*.

(b) Now set $g_n(x) = \sum_{k=1}^n k^{-1} f_k(x)$. Show that in $L^2([0, 2\pi])$ we have $\|g_n\|_2^2 = \pi \sum_{k=1}^n k^{-2}$.

(c) Show there exists a function $g \in L^2[0, 2\pi]$ such that $g_n \to g$ in $L^2([0, 2\pi])$ as $n \to \infty$.

(a) Note that for any integer $k \neq 0$, $\int_0^{2\pi} \cos(kx)dx = k^{-1}[\sin(kx)]_0^{2\pi} = 0$. Also for $n, m \in \mathbb{N}$, by the hint, $\sin(nx)\sin(mx) = (1/2)(\cos(nx - mx) - \cos(nx + mx))$, and hence

$$\int_0^{2\pi} f_n(x)f_m(x)dx = (1/2)\int_0^{2\pi} [\cos((n - m)x) - \cos((n + m)x)]dx = \begin{cases} \pi & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$$

(b) For $f, h \in L^2([0, 2\pi])$, write $\langle f, h\rangle$ for $\int_0^{2\pi} f(x)h(x)dx$. Then

$$\|g_n\|_2^2 = \langle g_n, g_n\rangle = \langle \sum_{i=1}^n i^{-1} f_i, \sum_{j=1}^n j^{-1} f_j\rangle.$$

By Question 59, $\langle f, h\rangle$ is linear in h, and since it is symmetric in h and f, also $\langle f, h\rangle$ is linear in f. That is, $\langle f, h\rangle$ is bilinear in f and h. Using this bilinearity and Part (a), we have

$$\|g_n\|_2^2 = \sum_{i=1}^n \sum_{j=1}^n i^{-1} j^{-1} \langle f_i, f_j\rangle = \sum_{i=1}^n i^{-2}\pi,$$

as required.

(c) For $m, n \in \mathbb{N}$ with $m < n$, similarly to Part (b) we have

$$\|g_n - g_m\|_2^2 = \langle \sum_{i=m+1}^{n} i^{-1} f_i, \sum_{j=m+1}^{n} j^{-1} f_i, \rangle = \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} i^{-1} j^{-1} \langle f_i, f_j \rangle = \sum_{i=m+1}^{n} i^{-2} \pi$$

which tends to zero as $m, n \to \infty$, since $\sum_{i=1}^{\infty} i^{-2} < \infty$.
Therefore $(g_n)_{n \geq 1}$ is a Cauchy sequence in L^2 so there exists $g \in L^2([0, 2\pi])$ with $g_n \to g$ in L^2 as $n \to \infty$, i.e. with $\|g_n - g\|_2 \to 0$ as $n \to \infty$.