49. Let $-\infty < a < b < \infty$. Suppose $g : [a, b] \to \mathbb{R}$ is a continuously differentiable, strictly increasing function. Show that for all bounded Borel-measurable $f : (a, b] \to \mathbb{R}$ we have the change of variables formula $\int_{g(a)}^{g(b)} f(y)dy = \int_a^b f(g(x))g'(x)dx$.

    Hint: First verify this for $f = \mathbf{1}_{(g(a),t]}$ with $g(a) < t \le g(b)$. Then use the Monotone Class theorem.

50. * (a) Show that $\{(x, y) \in \mathbb{R}^2 : x < y\} \in \mathcal{B} \otimes \mathcal{B}$.

    (b) Let $c \in (0, \infty)$. Show that $\{(x, y) \in \mathbb{R}^2 : x < y \le x + c\} \in \mathcal{B} \otimes \mathcal{B}$.

    (c) Suppose μ is a probability measure on $(\mathbb{R}, \mathcal{B})$. For $x \in \mathbb{R}$, let $F(x) = \mu((-\infty, x])$.

    Let $c \in \mathbb{R}$. Use Fubini's Theorem to show that $\int_{-\infty}^{\infty}(F(x + c) - F(x))dx = c$.

51. * For $d \in \mathbb{N}$ let λ_d denote d-dimensional Lebesgue measure.

    (a) Show that λ_2 and $\lambda_1 \otimes \lambda_1$ are the same measure on $(\mathbb{R}^2, \mathcal{B}_2)$.

    (b) Let $A \subset \mathbb{R}^2$ be a Borel set, and for $x \in \mathbb{R}$ let $A_x := \{y \in \mathbb{R} : (x, y) \in A\}$. Show that

$$\lambda_2(A) = \int_{-\infty}^{\infty} \lambda_1(A_x)dx,$$

52. * For $A \subset \mathbb{R}^d$ and $u \in \mathbb{R}^d$ let $A + u := \{a + u : a \in A\}$. Also if $d = 2$, for $x \in \mathbb{R}$ set $A_x := \{y \in \mathbb{R} : (x, y) \in A\}$.

    (a) Let $-\infty < a < b < \infty$, and let $I = (a, b)$. Let $y \in (0, \infty)$. Compute $\lambda_1((I + y) \setminus I)$.

    (b) Let $B \subset [0, 1]^2$ and suppose B is open (see Question 26) and B is convex, i.e. for all $u, v \in B$ and $\alpha \in (0, 1)$ we have $\alpha u + (1 - \alpha)v \in B$. Let e be the unit vector $(0, 1)$ and for $t > 0$ let $B(t) := B + te$. Given $x \in \mathbb{R}$, show that $B(t)_x = B_x + t$.

    (c) Let B be as in Part (b). Show that $\lambda_1((B(t) \setminus B)_x) = \min(t, \lambda_1(B_x))$,

    (d) Let B be as in Part (b). Show that $\lambda_2(B(t) \setminus B) \le t$.

    (e) Let B be as in Part (b). Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ denote projection onto the first co-ordinate, i.e. for $(x, y) \in \mathbb{R}^2$ se $\pi((x, y)) = x$. Show that $t^{-1}\lambda_2(B(t) \setminus B) \to \lambda_1(\pi_2(B))$ as $t \downarrow 0$.

    [The hint for Question 45 is also relevant here.]

53. Let $(X, \mathcal{M})$ be a measurable space and suppose $f : X \to [0, \infty]$ and $g : X \to [0, \infty]$ are Borel functions. Show that

$$\int_0^{\infty} \int_0^{\infty} \mu(\{x \in X : f(x) > s, g(x) > t\})dsdt = \int_X f(x)g(x)\mu(dx).$$

54. * (a) Let $\alpha \in \mathbb{R}$ be a fixed constant. Let $f(x) = x^\alpha$ for $x \in (0, 1]$. Determine the values of $p \in [1, \infty)$ (depending on α), such that $f \in L^p([0, 1])$.

    (b) Let $\alpha \in \mathbb{R}$, and let $g(x) = x^\alpha$ for $x \in [1, \infty)$. Determine the values of $p \in [1, \infty)$ (depending on α) such that $g \in L^p([1, \infty))$.

    [In this question, for any interval $I \subset \mathbb{R}$ we write $L^p(I)$ for $L^p(\mu)$ (in the sense of Definition 12.3) with μ taken to be Lebesgue measure on the space $X = I$ with the Borel σ-algebra.]