44. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $F_n \subset X$ with $F_n \in \mathcal{M}$ and $\mu(F_n) < \infty$, $\forall n \in \mathbb{N}$. Suppose also that $\mathcal{D} \subset \mathcal{M}$ is a π-system in X with $F_n \in \mathcal{D}$ for all $n \in \mathbb{N}$, and ν is a measure on $(X, \mathcal{M})$ such that $\nu(A) = \mu(A)$ for all $A \in \mathcal{D}$.

    (a) For $n \in \mathbb{N}$ set $E_n := \cup_{j=1}^n F_j$. Use the inclusion-exclusion formula from Question 39 to show for all $n \in \mathbb{N}, A \in \mathcal{D}$ that

    $$\mu(E_n) = \nu(E_n); \qquad \mu(A \cap E_n) = \nu(A \cap E_n).$$

    (b) Now suppose moreover that $\cup_{n=1}^\infty F_n = X$. Show that $\mu(A) = \nu(A)$ for all $A \in \sigma(\mathcal{D})$.
    This is the Uniqueness lemma (Theorem 5.7). It was proved in the notes under the extra assumption that $F_n \subset F_{n+1}$ for all $n \in \mathbb{N}$. You are asked here to prove it without this extra assumption.

45. * Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. Let $f : \Omega \to [0, \infty]$ be measurable, i.e. f is a nonnegative random variable. For $t \geq 0$ define $L(t) := \int_\Omega e^{-tf(\omega)} \mu(d\omega)$ (the *Laplace transform* of f).

    (a) Show that $\lim_{t\to\infty} L(t) = \mu(\{\omega \in \Omega : f(\omega) = 0\})$. Here we make the convention that $e^{-\infty} = 0$.

    (b) Show that $\lim_{t\downarrow 0} L(t) = \mu(\{\omega \in \Omega : f(\omega) < \infty\})$.

    (c) Show that $\lim_{t\downarrow 0}(t^{-1}(L(0) - L(t))) = \int f d\mu$ if the integral on the right is finite. *[Hint: use the fact that $1 - e^{-x} \leq x$ for $x \geq 0$.]* What about if the integral is infinite?

    [Hint: Given $f : (0, \infty) \to \mathbb{R}$ and $a \in \mathbb{R}$, if $f(t_n) \to a$ for any sequence $(t_n)_{n\geq 1}$ with $t_n \uparrow \infty$ as $n \to \infty$ then $f(t) \to a$ as $t \to \infty$. If $f(t_n) \to a$ for any sequence $(t_n)_{n\geq 1}$ with $t_n \downarrow 0$ as $n \to \infty$ then $f(t) \to a$ as $t \downarrow 0$.

46. * Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Show the following:

    (a) If $f : X \to [-\infty, \infty]$ is measurable, $E \in \mathcal{M}$, $\int_E |f| \, d\mu = 0$, then $f = 0$ a.e. on E.

    (b) If $f \in L^1(\mu)$ with $\int_E f \, d\mu = 0$ for all $E \in \mathcal{M}$, then $f = 0$ a.e. on X.

    (c) If $f \in L^1(\mu)$ with $\left|\int_X f \, d\mu\right| = \int_X |f| \, d\mu$, then either $f \geq 0$ a.e. on X, or $f \leq 0$ a.e. on X.

    (d) If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are measurable functions, then $\{x \in X : f(x) \neq g(x)\} \in \mathcal{M}$.

47. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable. Suppose $\{h_n\}_{n\geq 1}$ is a sequence in $\mathbb{R}$ such that $h_n \to 0$.

    (a) Show that for any $K \in (0, \infty)$ we have $\int_{-K}^K |f(x + h_n) - f(x)| dx \to 0$ as $n \to \infty$. [Hint: first suppose f is continuous, recalling that any continuous real-valued function on a compact interval is bounded. For general f, use Question 43]

    (b) Show that $\int_{-\infty}^\infty |f(x + h_n) - f(x)| dx \to 0$ as $n \to \infty$.

48. * Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Suppose $f, f_1, f_2, \ldots \in \mathbb{R}(X)$ such that $f_n \uparrow f$ pointwise and moreover $f_n \in L^1(\mu)$ and $\sup_n \int f_n d\mu < \infty$. Show that $f \in L^1(\mu)$ and $\int f_n d\mu \to \int f d\mu$ as $n \to \infty$.