41. * Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Suppose $F_1, \ldots, F_n$ are subsets of X with $F_i \in \mathcal{M}$ and $\mu(F_i) < \infty$ for each $i \in [n]$, where we set $[n] := \{1, \ldots, n\}$. For $S \subset [n]$ let $|S|$ denote the number of elemeents of S. Use the linearity of integration, and the fact that $\mu(A) = \int_X \mathbf{1}_A$ for any $A \in \mathcal{M}$, to prove the inclusion-exclusion formula from Question 14, namely

$$\mu(\cup_{i=1}^n F_i) = \sum_{J \in S(n)} (-1)^{|J|+1} \mu(\cap_{j \in J} F_j), \qquad \text{where } S(n) := \{J \subset \{1, \ldots, n\} : J \neq \varnothing\}.$$

[Hint: for any sets $G_1, \ldots, G_k \in \mathcal{M}$ we have $\mathbf{1}_{\cap_{i=1}^k G_i} = \prod_{i=1}^k \mathbf{1}_{G_i}$.]

42. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Suppose $f, g, h \in L^1(\mu)$.

    (a) For $F \in L^1(\mu)$ set $\|F\|_1 := \int |f| d\mu$. Show that $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$.

    (b) Show that $f - h \in L^1(\mu)$ and $h - g \in L^1(\mu)$ and $\|f - g\|_1 \leq \|f - h\|_1 + \|h - g\|_1$.

43. A function $f : \mathbb{R} \to \mathbb{R}$ is said is said to have **bounded support** if there exists $n \in \mathbb{N}$ such that $f(x) = 0$ whenever $|x| > n$. A function $g : \mathbb{R} \to \mathbb{R}$ is called a **step function** if we can write $g = \sum_{i=1}^k c_i \mathbf{1}_{I_i}$ for some $k \in \mathbb{N}$, $(c_1, \ldots, c_k) \in \mathbb{R}^k$ and $I_1, \ldots, I_k$ intervals in $\mathbb{R}$.

    Suppose $f : \mathbb{R} \to [0, \infty)$ is simple and has bounded support. Let $\varepsilon > 0$. Show that there exists a step function $g : \mathbb{R} \to \mathbb{R}$ such that $\int_{-\infty}^{\infty} |g - f| dx < \varepsilon$. *Hint: Recall Questions 23 and 15.*

44. * Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. Let $f : \Omega \to [0, \infty]$ be measurable, i.e. f is a nonnegative random variable. For $t \geq 0$ define $L(t) := \int_\Omega e^{-tf(\omega)} \mu(d\omega)$ (the *Laplace transform* of f).

    (a) Show that $\lim_{t \to \infty} L(t) = \mu(\{\omega \in \Omega : f(\omega) = 0\})$. Here we make the convention that $e^{-\infty} = 0$.

    (b) Show that $\lim_{t \downarrow 0} L(t) = \mu(\{\omega \in \Omega : f(\omega) < \infty\})$.

    (c) Show that $\lim_{t \downarrow 0} (t^{-1}(L(0) - L(t))) = \int f d\mu$ if the integral on the right is finite. *[Hint: use the fact that $1 - e^{-x} \leq x$ for $x \geq 0$.]* What about if the integral is infinite?

    [Hint: Given $f : (0, \infty) \to \mathbb{R}$ and $a \in \mathbb{R}$, if $f(t_n) \to a$ for any sequence $(t_n)_{n \geq 1}$ with $t_n \uparrow \infty$ as $n \to \infty$ then $f(t) \to a$ as $t \to \infty$. If $f(t_n) \to a$ for any sequence $(t_n)_{n \geq 1}$ with $t_n \downarrow 0$ as $n \to \infty$ then $f(t) \to a$ as $t \downarrow 0$.

45. * Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Show the following:

    (a) If $f : X \to [-\infty, \infty]$ is measurable, $E \in \mathcal{M}$, $\int_E |f| \, d\mu = 0$, then $f = 0$ a.e. on E.

    (b) If $f \in L^1(\mu)$ with $\int_E f \, d\mu = 0$ for all $E \in \mathcal{M}$, then $f = 0$ a.e. on X.

    (c) If $f \in L^1(\mu)$ with $\left|\int_X f \, d\mu\right| = \int_X |f| \, d\mu$, then either $f \geq 0$ a.e. on X, or $f \leq 0$ a.e. on X.

    (d) If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are measurable functions, then $\{x \in X : f(x) \neq g(x)\} \in \mathcal{M}$.

46. Suppose $f : \mathbb{R} \to \mathbb{R}$ is in L^1. Let $\varepsilon > 0$. Using Question 43, show there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $\|f - g\|_1 < \varepsilon$, i.e. $\int_{-\infty}^{\infty} |f(x) - g(x)| dx < \varepsilon$.