37. * Suppose $g : \mathbb{R} \to \mathbb{R}$ is integrable (with respect to Lebesgue measure), and let $t \in \mathbb{R}$.

    (a) Show that $\int_{-\infty}^{\infty} g(x-t)dx = \int_{-\infty}^{\infty} g(x)dx$.

    (b) Deduce that (with g as in (a)) for any $a, b \in \mathbb{R}$ with $a < b$, $\int_{a+t}^{b+t} g(x-t)dx = \int_a^b g(x)dx$.

    [*Hint: For part (a), start with the case where g is nonnegative and simple. Another way to write the result in (a) is* $\int h d\lambda_1 = \int g d\lambda_1$, *where we set* $h(x) = g(x-t)$]

38. * Let μ be counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$.

    (a) Let $k \in \mathbb{N}$. Show that if $f : \mathbb{N} \to [0, \infty)$ with $f(n) = 0$ for all $n > k$, then $\int_{\mathbb{N}} f d\mu = \sum_{n=1}^{k} f(n)$.

    [Hint: f must be simple.]

    (b) Show that if $g : \mathbb{N} \to [0, \infty)$ then $\int_{\mathbb{N}} g d\mu = \sum_{n=1}^{\infty} g(n)$.

    [Hint: use the Monotone Convergence theorem.]

    (c) Suppose $h : \mathbb{N} \to \mathbb{R}$ with $\sum_{n=1}^{\infty} |h(n)| < \infty$. Show that $\int_{\mathbb{N}} h d\mu = \sum_{n=1}^{\infty} h(n)$.

39. * Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Suppose $F_1, \ldots, F_n$ are subsets of X with $F_i \in \mathcal{M}$ and $\mu(F_i) < \infty$ for each $i \in [n]$, where we set $[n] := \{1, \ldots, n\}$. For $S \in \mathcal{P}([n])$, i.e. $S \subset [n]$, let $|S|$ denote the number of elemeents of S. Use the linearity of integration, and the fact that $\mu(A) = \int_X \mathbf{1}_A$ for any $A \in \mathcal{M}$, to prove the *inclusion-exclusion formula*

$$\mu(\cup_{i=1}^{n} F_i) = \sum_{J \in \mathcal{P}([n]) \setminus \{\varnothing\}} (-1)^{|J|+1} \mu(\cap_{j \in J} F_j),$$

    [Hint: for any sets $G_1, \ldots, G_k \in \mathcal{M}$ *we have* $\mathbf{1}_{\cap_{i=1}^{k} G_i} = \prod_{i=1}^{k} \mathbf{1}_{G_i}$.*]*

40. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Suppose $f, g, h \in L^1(\mu)$.

    (a) For $F \in L^1(\mu)$ set $\|F\|_1 := \int |F| d\mu$. Show that $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$.

    (b) Show that $f - h \in L^1(\mu)$ and $h - g \in L^1(\mu)$ and $\|f - g\|_1 \leq \|f - h\|_1 + \|h - g\|_1$.

41. * A function $f : \mathbb{R} \to \mathbb{R}$ is said is said to have **bounded support** if there exists $n \in \mathbb{N}$ such that $f(x) = 0$ whenever $|x| > n$.

    Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable (with respect to Lebesgue measure). Let $\varepsilon > 0$. Show that there exists integrable $g : \mathbb{R} \to \mathbb{R}$ such that $\int_{-\infty}^{\infty} |f(x) - g(x)| dx < \varepsilon$, and g has bounded support

42. A function $g : \mathbb{R} \to \mathbb{R}$ is called a **step function** if we can write $g = \sum_{i=1}^{k} c_i \mathbf{1}_{I_i}$ for some $k \in \mathbb{N}$, $(c_1, \ldots, c_k) \in \mathbb{R}^k$ and $I_1, \ldots, I_k$ intervals in $\mathbb{R}$.

    Suppose $f : \mathbb{R} \to [0, \infty)$ is simple and has bounded support (i.e., there exists $n \in \mathbb{N}$ with $f(x) = 0$ whenever $|x| > n$). Let $\varepsilon > 0$. Show that there exists a step function $g : \mathbb{R} \to \mathbb{R}$ such that $\int_{-\infty}^{\infty} |g - f| dx < \varepsilon$. *Hint: Recall Questions 17 and 23.*

43. Suppose $f : \mathbb{R} \to \mathbb{R}$ is in L^1. Let $\varepsilon > 0$. Using Question 42, show there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $\|f - g\|_1 < \varepsilon$, i.e. $\int_{-\infty}^{\infty} |f(x) - g(x)| dx < \varepsilon$.