31. * (a) Let $(X, \mathcal{M})$ be a measurable space, and let $f_n : X \to \mathbb{R}$ be measurable functions. Show that the set of points
$$\{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists in } \mathbb{R}\}$$
is in $\mathcal{M}$.
(b) Taking $(\Omega, \mathcal{F}, \mathbb{P})$ to be a probability space, and random variables (i.e., measurable functions) $Y_1, Y_2, \ldots : \Omega \to \mathbb{R}$ show that for any constant $\mu \in \mathbb{R}$ the set:
$$\left\{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} Y_i(\omega) = \mu\right\}$$
is in $\mathcal{F}$. Deduce that expressions like $\mathbb{P}[\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} Y_i = \mu]$ are meaningful.

32. Let $(X, \mathcal{M})$ be a measurable space.
(a) Show that if $E \in \mathcal{M}$, then its indicator function $\mathbf{1}_E$ defined by $\mathbf{1}_E(x) = 1$ for $x \in E$ and $\mathbf{1}_E(x) = 0$ for $x \notin E$, is a measurable function.
(b) Let $f : X \to \mathbb{R}$ be function with finite range $f(X) = \{\alpha_1, \ldots, \alpha_n\}$ (with $\alpha_1, \ldots, \alpha_n$ distinct), so that $f = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{A_i}$, where $A_i = \{x \in X : f(x) = \alpha_i\}$. Show that f is measurable if and only if $A_1, \ldots, A_n \in \mathcal{M}$.

33. * Suppose $(X, \mathcal{M}, \mu)$ is a σ-finite measure space and $f : X \to [0, \infty]$ is measurable.

    (a) Prove that if $a \in (0, \infty)$ then $\mu(f^{-1}[a, \infty]) \leq a^{-1} \int f d\mu$. [When μ is a probability measure, this is called *Markov's inequality*]

    (b) Prove that if $\int f d\mu = 0$, then $\mu(f^{-1}((0, \infty])) = 0$.

34. * Let $(X, \mathcal{M})$ be a measurable space. Suppose $f : X \to [0, \infty)$ and $g : X \to [0, \infty)$ are measurable functions. Define the set $A \subset X \times \mathbb{R} \times \mathbb{R}$ by $A := \{(x, s, t) : f(x) > s, g(x) > t\}$. Let $\mathcal{B}$ denote the Borel σ-algebra in $\mathbb{R}$. Show that $A \in \mathcal{M} \otimes \mathcal{B} \otimes \mathcal{B}$, where $\mathcal{M} \otimes \mathcal{B} \otimes \mathcal{B}$ is the σ-algebra generated by the collection of all sets in $X \times \mathbb{R} \times \mathbb{R}$ of the form $B \times C \times D$ with $B \in \mathcal{M}$, $C \in \mathcal{B}$ and $D \in \mathcal{B}$.

    [Hint: You can use a similar approach to the proof of Theorem 10.13]

35. * (a) Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. Show that that for all $A \subset X \times Y$ with $A \in \mathcal{M} \otimes \mathcal{N}$, and all $y \in Y$, the horizontal cross-section $A_{[y]}$ of A defined by
$$A_{[y]} := \{x \in X : (x, y) \in A\}$$
satisfies $A_{[y]} \in \mathcal{M}$. *[Hint: First show the class of $A \subset X \times Y$ with $A_{[y]} \in \mathcal{M}$ is a σ-algebra]*
(b) Suppose $f : X \to [0, \infty]$ is such that $\mathrm{hyp}(f) \in \mathcal{M} \otimes \mathcal{B}$. Show that f is a measurable function.

36. Let $W \in \mathcal{B}$ (the Borel sets in $\mathbb{R}$) with $W \neq \varnothing$. Recall from Definition 10.3 that $\mathcal{B}_W := \{B \subset W : B \in \mathcal{B}\}$,

    (a) Show that $\mathcal{B}_W = \{A \cap W : A \in \mathcal{B}\}$.

    (b) Show that $\mathcal{B}_W$ is the σ-algebra (in W) generated by the collection of all sets of the form $(-\infty, a] \cap W$ with $a \in \mathbb{R}$.