26. * (a) Show that if $U \subset \mathbb{R}^2$ is open and $x \in U$, then we can find a rectangle $R \in \mathcal{R}_2$ with corners having rational coordinates such that $x \in R \subset U$. *[We say that a set $A \subset \mathbb{R}^2$ is* open *if for every $x \in A$ there is a disk of positive radius centred on x that is contained in A.]*

    (b) Show that $\sigma(\mathcal{O}_2) = \mathcal{B}_2$, where $\mathcal{O}_2$ is the class of all open sets in $\mathbb{R}^2$, and $\mathcal{B}_2$ is the Borel σ-algebra in $\mathbb{R}^2$ (see Definition 8.1).

27. * Suppose ρ is a rotation on $\mathbb{R}^2$, i.e. pre-multiplication by a 2×2 matrix M with $M^T = M^{-1}$ (viewing elements of $\mathbb{R}^2$ as column vectors). Let λ_2 denote 2-dimensional Lebesgue measure (see Definition 8.10).

    (a) Show that $|\rho(x)| = |x|$ for all $x \in \mathbb{R}^2$, where for $x = (x_1, x_2)' \in \mathbb{R}^2$ we put $|x| = \sqrt{x_1^2 + x_2^2}$.

    (b) Show that $\rho(A) \in \mathcal{B}_2$ for all $A \in \mathcal{B}_2$.

    (c) Define a measure μ on $\mathcal{B}_2$ by $\mu(A) = \lambda_2(\rho(A))$ for all $A \in \mathcal{B}_2$. Show that μ is translation invariant, i.e. $\mu(A + x) = \mu(A)$ for all $A \in \mathcal{B}_2$ and all $x \in \mathbb{R}^2$.

    (d) Show that the measure λ_2 is rotation invariant, i.e. $\lambda_2(\rho(A)) = \lambda_2(A)$ for all Borel $A \subset \mathbb{R}^2$ (and for any rotation ρ).

    You may use without proof the fact that every translation-invariant measure ν on $(\mathbb{R}^2, \mathcal{B}_2)$ is of the form $\nu = c \times \lambda_2$ for some constant c.

28. (a) Show that $\lambda_2(L) = 0$ for any line segment $L \subset \mathbb{R}^2$. *[You may use the result from Question 27 without proof].*

    (b) Let $r > 0$ and set $D := \{x \in \mathbb{R}^2 : |x| < r\}$, the open disk of radius r in $\mathbb{R}^2$ centred on the origin (we define $|x|$ as in the previous question). By approximating to D by an increasing sequence of regular polygons contained in D, show that $\lambda_2(D) = \pi r^2$.

    You may use without proof the 'half base times height' formula for the Lebesgue measure (area) of a triangle. You may also use without proof the fact that $(\sin x)/x \to 1$ as $x \downarrow 0$.

29. Suppose F is a function with the properties assumed in Exercise 25.

    (a) Prove that there is a unique measure μ_F on $(\mathbb{R}, \mathcal{B})$ with the property that $\mu_F((a, b]) = F(b) - F(a)$ for all $a, b \in \mathbb{R}$ with $a < b$. (You may assume without proof Carathéodory's extension theorem, along with the results of Exercise 25.)

    (b) Given $y \in \mathbb{R}$, show that the μ_F-measure of the one-point set $\{y\}$ is $\mu_F(\{y\}) = F(y) - F(y-)$, where $F(y-) = \lim_{z \uparrow y} F(z)$.

    (c) Show that $\mu_F([a, b]) = F(b) - F(a-)$, and also find the formulas for $\mu_F((a, b))$ and $\mu_F([a, b))$, when $-\infty < a < b < \infty$.

    Remark. The measure μ_F is called the **Lebesgue-Stieltjes measure** corresponding to the function F.

30. * Prove that if $W \subset \mathbb{R}$ is a Borel set, and $f : W \to \mathbb{R}$ is an increasing function (i.e. $f(x) \le f(y)$ whenever $x, y \in W$ with $x < y$), then f is Borel-measurable.