On this sheet, λ_1 denotes the Lebesgue measure on $(\mathbb{R}, \mathcal{B})$.

21. * Show that λ_1 has the scaling property: for any real number $c \neq 0$ and any Borel set $B \in \mathcal{B}$, we have $\lambda_1(cB) = |c|\lambda_1(B)$. Here cB is defined to be the set $\{cx : x \in B\}$.

22. * Suppose μ is a translation invariant measure on $(\mathbb{R}, \mathcal{B})$. Set $\gamma := \mu((0,1])$ and assume $0 < \gamma < \infty$.
    (a) Show that $\mu((0, 1/n]) = \gamma/n$ for all $n \in \mathbb{N}$.
    (b) Show that $\mu((0, q]) = \gamma q$ for all rational $q > 0$.
    (c) Let $\mathcal{I}'$ be the class of half-open intervals in $\mathbb{R}$ with rational endpoints, i.e. the class of intervals of the form $(q, r]$ with $q \in \mathbb{Q}$, $r \in \mathbb{Q}$ and $q \leq r$. Show that $\mu(I) = \gamma\lambda_1(I)$ for all $I \in \mathcal{I}'$.
    (d) Show that $\sigma(\mathcal{I}') = \mathcal{B}$. You may use without proof the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, that is, every non-empty open interval in $\mathbb{R}$ contains at least one rational number.
    (e) Use the Uniqueness lemma to show that $\mu(B) = \gamma\lambda_1(B)$ for all $B \in \mathcal{B}$.

23. * Suppose X is a non-empty set and $\mathcal{S}$ is a semi-algebra in X. As in Chapter 6 of the notes, let $\mathcal{U}$ be the class of sets of the form $\cup_{i=1}^k A_i$ with $k \in \mathbb{N}$ and $A_1, \ldots, A_k$ pairwise disjoint sets in $\mathcal{S}$.

    (a) Show by induction on k that if $A \in \mathcal{U}$ then $A^c \in \mathcal{U}$, i.e. $\mathcal{U}$ is closed under complementation.

    (b) Show also that $\mathcal{U}$ is closed under pairwise intersections and deduce that $\mathcal{U}$ is an algebra.

    (c) Deduce that $\mathcal{U}$ is the algebra generated by $\mathcal{S}$. (Generated algebras are defined analogously to generated σ-algebras. Write $\mathcal{A}(\mathcal{S})$ for the algebra generated by $\mathcal{S}$.)

24. Suppose X is a non-empty set, $\mathcal{S}$ is a semi-algebra in X and π is a pre-measure on $(X, \mathcal{S})$.
    (a) Show that if $A, A_1, \ldots, A_k \in \mathcal{S}$ with $A_1, \ldots, A_k$ pairwise disjoint and $\cup_{i=1}^k A_i \subset A$, then $\sum_{i=1}^k \pi(A_i) \leq \pi(A)$.
    (b) Show that π is countably additive, i.e. $\pi(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \pi(A_n)$ whenever $A_1, A_2, \ldots \in \mathcal{S}$ are pairwise disjoint with $\cup_{n=1}^\infty A_n \in \mathcal{S}$.
    Hint: The result from Question 23 might be useful.

25. Let $F : (-\infty, \infty) \to \mathbb{R}$ be a non-decreasing, right continuous function (right continuity is defined in question 10).

    Let $\mathcal{I}$ denote the set of bounded half-open intervals in $\mathbb{R}$ (as in lectures). For $I \in \mathcal{I}$, put

    $$\lambda_F(I) = F(b) - F(a), \quad \text{where } I = (a, b], \quad \text{and } \lambda_F(\varnothing) = 0.$$

    (a) Check that $\lambda_F(I) \geq 0$ for all $I \in \mathcal{I}$.
    (b) Show that the set function λ_F is finitely sub-additive on $\mathcal{I}$, the class of bounded half-open intervals in $\mathbb{R}$. That is, show that if $A, A_1, A_2, \ldots A_n \in \mathcal{I}$ with $A \subset \cup_{i=1}^n A_i$ then $\lambda_F(A) \leq \sum_{i=1}^n \lambda_F(A_i)$.
    (c) Show that λ_F is finitely additive on $\mathcal{I}$. That is, show that if $A_1, A_2, \ldots A_n \in \mathcal{I}$ are pairwise disjoint with $A := \cup_{i=1}^n A_i \in \mathcal{I}$, then $\lambda_F(A) = \sum_{i=1}^n \lambda_F(A_i)$.
    (d) Show that λ_F is countably sub-additive on $\mathcal{I}$. That is, show that if $A, A_1, A_2, \ldots \in \mathcal{I}$ with $A \subset \cup_{i=1}^\infty A_i$ then $\lambda_F(A) \leq \sum_{i=1}^\infty \lambda_F(A_i)$.

    [Hint: For (b) and (c), adapt the proof of Lemma 4.1. For (d), adapt the proof of Theorem 4.2.]