55. * Let $p \in [1, \infty)$ and let $f \in L^p(\mathbb{R})$. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be real-valued sequences such that $\sum_{n=1}^{\infty} |a_n| < \infty$. Show that the sequence of functions $f_n(x) := \sum_{k=1}^{n} a_k f(x - b_k)$ converges in $L^p(\mathbb{R})$.

56. * Suppose $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are sequences of nonnegative numbers, such that $A := \sum_{n=1}^{\infty} a_n^{4/3} < \infty$ and $B := \sum_{n=1}^{\infty} b_n^4 < \infty$. Show that $\sum_{n=1}^{\infty} a_n b_n \leq A^{3/4} B^{1/4}$.

57. Suppose that $(X, \mathcal{M}, \mu)$ is a σ-finite measure space, and $1 \leq p < q < \infty$.

    (a) Show that if μ is a probability measure and $f \in L^q(\mu)$, then $\|f\|_p \leq \|f\|_q$.
    [Hint: note that $f = f \cdot 1$, and apply Hölder's inequality]

    (b) Show that if $\mu(X) < \infty$ then $L^q(\mu) \subset L^p(\mu)$.

    (c) Give an example to show that if $\mu(X) = \infty$, then we might not have $L^q(\mu) \subset L^p(\mu)$.

58. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Let $p \in (1, \infty)$. Suppose $f \in \mathbb{R}(X)$ and (for all $n \in \mathbb{N}$) $f_n \in \mathbb{R}(X)$, with $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$. For all $n \in \mathbb{N}$ and $x \in X$, set

$$g_n(x) = \sum_{k=1}^{n} |f_k(x)| \quad \text{and} \quad g_\infty(x) = \sum_{k=1}^{\infty} |f_k(x)|.$$

    (i) Show that $\|g_n\|_p \to \|g_\infty\|_p$ as $n \to \infty$, and deduce that $\|g_\infty\|_p < \infty$.

    (ii) Show that the function $h(x) := \sum_{n=1}^{\infty} f_n(x)$ is well-defined and finite μ-a.e., that is, the sum converges for μ-a.e. $x \in X$.

59. * Let $W \in \mathcal{B}$, and for $f, g \in L^2(W)$, write $\langle f, g\rangle = \int_W f(x)g(x)dx$. Show that if also $h \in L^2(W)$ and $a, b \in \mathbb{R}$ then $\langle f, ag + bh\rangle = a\langle f, g\rangle + b\langle f, h\rangle$.

60. * For $n \in \mathbb{N}$, let $f_n(x) = \sin(nx)$.

    (a) Show that for $n, m \in \mathbb{N}$ with $n \neq m$ we have $\int_0^{2\pi} f_n(x) f_m(x)dx = 0$, while $\int_0^{2\pi} (f_n(x))^2 dx = \pi$.
    [Hint: recall that $\cos(a + b) = \cos a \cos b - \sin a \sin b$*].*

    (b) Now set $g_n(x) = \sum_{k=1}^{n} k^{-1} f_k(x)$. Show that in $L^2([0, 2\pi])$ we have $\|g_n\|_2^2 = \pi \sum_{k=1}^{n} k^{-2}$.

    (c) Show there exists a function $g \in L^2[0, 2\pi]$ such that $g_n \to g$ in $L^2([0, 2\pi])$ as $n \to \infty$.