9.3 Delta hedging with futures. Another way to achieve delta neutrality of a portfolio is to use a futures contract on the underlying asset. Assume that an asset does not generate any dividends. Consider one futures contract on one unit of the underlying asset. Suppose that the contract expiration date is T. If the current asset price is $S(0) = S$, then the futures price is

$$F(0,T) = F(0,T,S) = Se^{rT}$$

The Delta for the futures contract is

$$\Delta = \frac{\partial F(0,T,S)}{\partial S} = e^{rT}.$$

If the asset price instantly changes by δS, then a holder of a long futures gains/loses $\delta F(0,T,S) \approx e^{rT}\delta S$ (this amount is added/subtracted to/from the margin account).

Assume we have got a portfolio formed by m identical options: with maturity T, strike price K and with delta Δ_c. What position in the futures contract with maturity T' (on the same underlying) should be added to achieve delta neutrality?

Solution. The portfolio delta is $\Delta_c m$. The futures contract on one unit of the underlying has delta $e^{rT'}$. Therefore, to achieve delta neutrality, a futures contract on m_f units of the underlying asset is needed, m_f solves the equation

$$e^{rT'}m_f + \Delta_c m = 0$$

and hence, $m_f = -\Delta_c m e^{-rT'}$.

If $m > 0$, then the position in options is long and we need a short position in futures. If $m < 0$, then the position in options is short and we need a long position in futures.

Note that the futures contracts can have different maturity from the options.

9.4 Gamma (Γ). The gamma of a financial derivative is defined as the rate of change of the derivative delta with respect to the price of the underlying, when all else remains the same. So gamma is the second (mathematical) partial derivative of the (financial) derivative price with respect to the underlying price and is mathematically expressed as:

$$\Gamma = \frac{\partial^2 f}{\partial S^2} = \frac{\partial \Delta}{\partial S}.$$

For example, $\Gamma = 0.05$ means, by Taylor expansion, that

$$\Delta(S + \delta S) \approx \Delta(S) + 0.05 \times \delta S + \dots$$

Γ is a measure of sensitivity of Δ to changes of the underlying price. It can be positive or negative (or zero).

- If Γ is small, Δ only changes slowly and in order to keep the portfolio delta-neutral, adjustments to the portfolio can be made less frequently.
- If Γ is large, i.e. Δ is very sensitive to the underlying price, the portfolio will need to be adjusted frequently to maintain delta-neutrality.
- Γ *of a call option* with a strike price K and time T to expiration. Recall that for this option we have

$$\Delta = \frac{\partial C}{\partial S} = \Phi(x_1), \quad x_1 = \frac{\log S - \log K + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

so differentiating with respect to S (and using the fundamental theorem of calculus and the chain rule) gives a Gamma for this option of

$$\begin{aligned}\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta}{\partial S} &= \varphi(x_1)\frac{\partial x_1}{\partial S} \\ &= \frac{e^{-x_1^2/2}}{\sqrt{2\pi}}\frac{1}{S\sigma\sqrt{T}}.\end{aligned}$$

- The Gamma for the corresponding put option is just the same as the gamma for the call option, since $\Delta_{\text{put}} = \Delta_{\text{call}} - 1$ so that

$$\Gamma_{\text{put}} = \frac{\partial}{\partial S}\Delta_{\text{put}} = \frac{\partial}{\partial S}\Delta_{\text{call}} = \Gamma_{\text{call}}.$$

- Γ of the underlying asset is always 0, since for the asset $\Delta = 1$ so $\Gamma = \frac{\partial \Delta}{\partial S} = 0$.

9.5 Gamma neutrality and gamma-delta neutrality

A portfolio is said to be **gamma-neutral** if its gamma is zero. As discussed in the preceding section, if a portfolio is kept both delta-neutral and gamma-neutral, then the intervals between rebalancing to maintain delta-neutrality do not need to be so short.

We now argue further that gamma-neutrality provides protection against large fluctuations in the stock price between rebalancing instances (whereas delta-neutrality protects against small fluctuations).

Assume that the volatility is constant. Then by Taylor expansion as in Section 9.0,

$$\delta f \approx \frac{\partial f}{\partial S}\delta S + \frac{\partial f}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(\delta S)^2$$

and so for a delta-neutral portfolio (with $\Delta = \partial f/\partial S = 0$),

$$\delta f \approx \Theta \delta t + \frac{1}{2}\Gamma(\delta S)^2$$

If $|\delta S|$ is large we cannot ignore the second term, especially if Γ is large. Note that the Ito formula suggests we could replace $(\delta S)^2$ by the deterministic quantity $\sigma^2 S^2 dt$, but this for an infinitesimally small time-increment and might not be so accurate for the actual time-increments in between portfolio rebalancing.

Assuming that $\Delta = \Gamma = 0$ and that σ is constant, the Taylor expansion simplifies to $\delta f \approx \Theta \delta t$, if terms of higher order are neglected. Therefore, over a short period of time the portfolio changes are predictable.

How do we achieve delta- and gamma neutrality in practice? Consider a portfolio formed by m_1 call options ($m_1 > 0$ means long options, $m_1 < 0$ means short options) with the same strike price and maturity. Let Δ_1 and Γ_1 be the delta and the gamma of each of these options. The portfolio delta is $\Delta_1 m_1$, the portfolio gamma is $\Gamma_1 m_1$.

- We would like to make the portfolio both delta- and gamma-neutral.
- Delta-neutrality can be achieved by adding a certain amount of the underlying asset. But a position in the underlying asset or in a forward/futures contract on the underlying has zero gamma.
- Therefore, we need another financial derivative to neutralize the portfolio gamma. Usually, other options are used for this purpose.

Assume that another type of traded options (calls or puts) is available and let Δ_2 and Γ_2 be the delta and the gamma of these options (assume $\Gamma_2 \neq \Gamma_1$). Add m_2 of these traded options to the portfolio, where m_2 is found from the equation

$$m_1 \times \Gamma_1 + m_2 \times \Gamma_2 = 0$$

(if $m_2 < 0$, then it means that we add a short position in these options). Adding m_2 of the traded options changes the portfolio delta by $\Delta_2 m_2$. Assume that we use a position in m_u units the underlying asset to get delta neutrality (this does not affect Γ since the underlying has a Γ of zero). Then m_u can be found from the following equation

$$m_1 \times \Delta_1 + m_u + m_2 \times \Delta_2 = 0.$$

Note that m_2 is already known. The portfolio obtained is both $\Delta-$neutral and $\Gamma-$neutral.

Example. Consider a portfolio which is delta-neutral and has $\Gamma = 1000$. How can we make the portfolio gamma-neutral if put options with $\Delta = -0.5$ and $\Gamma = 2$ are available? What adjustment will be needed to keep the portfolio to be delta-neutral as well?

Solution. First make the portfolio Γ-neutral by adding a a position in m_1 put options with $2m_1 + 1000 = 0$, so that $m_1 = -500$. A negative number of the options means that the position is short. This affects the portfolio delta; delta becomes equal to $-500 \times (-0.5) = 250$. Therefore a quantity -250 of the underlying asset (short position) has to be added to maintain delta-neutrality of the portfolio.

In general, a portfolio can be made gamma-neutral only instantaneously, as in the case of delta-neutrality. Periodic adjustments of the portfolio are needed to maintain both delta- and gamma-neutrality. A delta-gamma neutral portfolio is far less sensitive to changes in the stock price than a delta-neutral portfolio.

9.6 Vega ($\mathcal{V}$). The vega of a financial derivative is the rate of change of the value of the derivative with respect to the volatility of the underlying asset:

$$\mathcal{V} = \frac{\partial f}{\partial \sigma}.$$

The vega of a position in the underlying asset is zero since $\partial S/\partial\sigma = 0$.

For either a call or a put option with strike price K and time T to expiration, the vega is

$$\mathcal{V} = S\sqrt{T}\varphi(x_1) = S\sqrt{T}\frac{e^{-x_1^2}}{\sqrt{2\pi}}.$$

To derive this formula, as usual we start with the Black-Scholes formula $C = S\Phi(x_1) - Ke^{-rT}\Phi(x_0)$. So for the call,

$$\mathcal{V} = \frac{\partial C}{\partial \sigma} = S\varphi(x_1)\frac{\partial x_1}{\partial \sigma} - Ke^{-rT}\varphi(x_0)\frac{\partial x_0}{\partial \sigma}.$$

By equation (1) from the proof of the formula for Δ_{call} in Section 9.1, we have $S\varphi(x_1) = Ke^{-rT}\varphi(x_0)$, and therefore

$$\mathcal{V} = S\varphi(x_1)\left(\frac{\partial x_1}{\partial \sigma} - \frac{\partial x_0}{\partial \sigma}\right).$$

Now,

$$x_1 = \frac{\log(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\log(S/K) + rT}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

so that

$$\frac{\partial x_1}{\partial \sigma} = -\frac{\log(S/K) + rT}{\sigma^2\sqrt{T}} + \frac{\sqrt{T}}{2}$$

and similarly,

$$\frac{\partial x_0}{\partial \sigma} = -\frac{\log(S/K) + rT}{\sigma^2\sqrt{T}} - \frac{\sqrt{T}}{2}$$

and hence

$$\mathcal{V} = S\varphi(x_1)\left(\frac{\partial x_1}{\partial \sigma} - \frac{\partial x_0}{\partial \sigma}\right) = \sqrt{T}S\varphi(x_1)$$

as claimed. As for the put, we have by put-call parity that $P = C + Ke^{-rT} - S$, and differentiating the put-call parity with respect to σ shows that $\mathcal{V}_{\text{put}} = \mathcal{V}_{\text{call}}$, since $Ke^{-rT} - S$ does not depend on σ so its partial derivative with respect to σ is zero.

The value of the delta-gamma neutral portfolio can be very sensitive to changes in volatility. The aim of vega hedging is to remove volatility risk, to make the portfolio insensitive to small changes in volatility.

We are going to use the same method as before to construct delta-gamma-vega neutral portfolio. Therefore we need to add a third financial derivative to the portfolio.

Suppose we have got m_1 call options (long or short) with the same strike price and maturity. Let Δ_1, Γ_1 and $\mathcal{V}_1$ be the delta, the gamma and the vega of each of these options. We would like to make this position delta-gamma-vega neutral. We add

- a quantity m_u of the underlying asset
- m_2 of options II (calls or puts), each with Δ_2, Γ_2 and $\mathcal{V}_2$
- m_3 of options III (calls or puts), each with Δ_3, Γ_3 and $\mathcal{V}_3$

We have got three equations:

$$m_1 \times \mathcal{V}_1 + m_2 \times \mathcal{V}_2 + m_3 \times \mathcal{V}_3 = 0, \qquad \text{Vega neutrality}$$

$$m_1 \times \Gamma_1 + m_2 \times \Gamma_2 + m_3 \times \Gamma_3 = 0, \quad \text{Gamma neutrality}$$

$$m_1 \times \Delta_1 + m_2 \times \Delta_2 + m_3 \times \Delta_3 + m_u = 0, \quad \text{Delta neut.}$$

with three unknowns m_u, m_2 and m_3.

Example. Consider a portfolio Π with $\Delta_\Pi = 0, \Gamma_\Pi = -3000$ and $\mathcal{V}_\Pi = -5000$. Assume that two different types of options are available with risk parameters $\Delta_1 = 0.3, \Gamma_1 = 0.5, \mathcal{V}_1 = 2$ and $\Delta_2 = 0.2, \Gamma_2 = 0.6, \mathcal{V}_2 = 3$ respectively. How can the portfolio be made both gamma and vega neutral? What further adjustment is needed to restore delta neutrality?

Solution. Add m_1 options of the first type and m_2 of the second type. For gamma neutrality, we need

$$0 = -3000 + 0.5m_1 + 0.6m_2$$

and for vega neutrality, we need

$$0 = -5000 + 2m_1 + 3m_2.$$

Solving this system of equations (and rounding to the nearest integer), we get $m_2 = -1750/0.15 = -11667$ and $m_1 = 20000$. So, we need 20000 long options of the first type and 11667 short options of the second type. The portfolio delta becomes equal to

$$0.3 \times 20000 - 0.2 \times 11667 = 3667$$

Therefore we need to short sell a quantity 3667 of the underlying asset to restore delta neutrality.