9 Elements of risk management

There are five quantities representing the market sensitivities of a portfolio of financial derivatives. They are: $\Delta, \Gamma, \mathcal{V}, \Theta, \rho$ and often they are simply called the Greek letters or simply the Greeks (though $\mathcal{V}$ is not a Greek letter...). Each Greek letter measures a different aspect of the portfolio risk. Understanding Greeks helps to manage portfolio risks appropriately. Options and many other financial derivatives are volatile and risky instruments and any change in the variables used in the valuation model (time to expiration, the stock price, volatility and the interest rate) will affect the portfolio price.

Consider a portfolio Π of financial derivatives dependent on a single underlying asset whose price follows the log-normal model, e.g.,

- 1 short call option
- 1 long call option and 1 short put option with different strike prices and different expiration dates,
- 10 short call options and a long forward on 1200 units of the underlying asset
- 20 put options, a short futures on 500 units of the underlying asset, 1 million pounds invested at the risk-free interest rate.

Assuming that r and the volatility σ are constant, we can consider the portfolio value

$$f = f(t, S)$$

as a function depending on time t and the stock price $S = S(t)$. Denote

$$\text{Delta} : \Delta = \frac{\partial f}{\partial S}, \qquad \text{Theta} : \Theta = \frac{\partial f}{\partial t}, \qquad \text{Gamma} : \Gamma = \frac{\partial^2 f}{\partial S^2},$$

The partial derivative has the interpretation that we consider the changes of f given changes of a single variable with everything else being constant. By the Taylor expansion, as seen earlier, we can write

$$\begin{aligned} \delta f &= f(t+\delta t, S+\delta S) - f(t, S) \\ &= \Theta \times \delta t + \Delta \times \delta S + \frac{1}{2}\Gamma \times (\delta S)^2 + \dots \end{aligned}$$

More generally, we might not assume the volatility σ is constant. In that case, the value of the portfolio is $f = f(t, S, \sigma)$, a function depending on time t, the stock price S and the stock price volatility σ. In that case we should consider Vega: $\mathcal{V} = \frac{\partial f}{\partial \sigma}$, and we have the more general Taylor expansion

$$\begin{aligned} \delta f &= f(t+\delta t, S+\delta S, \sigma+\delta\sigma) - f(t, S, \sigma) \\ &= \Theta \times \delta t + \Delta \times \delta S + \mathcal{V} \times \delta\sigma + \frac{1}{2}\Gamma \times (\delta S)^2 + \dots \end{aligned}$$

9.1 Delta (Δ). The delta of a derivative security is defined as the rate of change of its price with respect to the price of the underlying asset.

$$\Delta = \frac{\partial f(t, S, \sigma)}{\partial S}$$

- Δ is an indicator of a risk relating to the change of the underlying price.
- If $\Delta > 0$, then a sudden increase of the underlying price will result in a positive change of the portfolio price. If $\Delta < 0$, then a sudden increase of the underlying price will result in a negative change of the portfolio price, etc.
- The delta of a European call option is between 0 and +1 (see below).
- The delta of a European put option is between 0 and −1 (see below).
- The delta of the underlying asset (1 unit) is always 1, since $\frac{\partial S}{\partial S} = 1$.

Recall the **fundamental theorem of calculus** which says that if $f(x) = \int_{-\infty}^{x} g(t)dt$ for a continuous function g, then $\frac{df}{dx} = g(x)$.

Recall also that $\Phi(x) = \int_{-\infty}^{x} \varphi(t)dt$ where $\varphi(t) = (2\pi)^{-1/2} \exp(-t^2/2)$, so by the fundamental theorem of calculus $\frac{d\Phi}{dx} = \varphi(x)$. Recall also the **chain rule** which says that if $f(x) = g(h(x))$ where g and h are differentiable functions, then writing $f'(x)$ for $\frac{df}{dx}$ we have

$$\frac{df}{dx} = \frac{dg(h(x))}{dx} = \frac{dg(h(x))}{dh(x)} \times \frac{dh(x)}{dx}, \qquad \text{i.e.} \quad f'(x) = g'(h(x))h'(x).$$

For example, if $f(x) = \sin(x^2)$ then $f'(x) = 2x\cos(x^2)$.

The delta of a European call option is given by

$$\Delta = \Phi(x_1).$$

To obtain this, we start from the Black-Scholes formula:

$$C = S\Phi(x_1) - Ke^{-rT}\Phi(x_0).$$

where

$$x_0 = \frac{\log(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}};$$
$$x_1 = x_0 + \sigma\sqrt{T} = \frac{\log(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}},$$

By the definition of Δ, in the case of the European call we have

$$\Delta = \frac{\partial C}{\partial S}$$
$$= \Phi(x_1) + S\varphi(x_1)\frac{\partial x_1}{\partial S} - Ke^{-rT}\varphi(x_0)\frac{\partial x_0}{\partial S}$$

so that since $\partial x_1/\partial S = \partial x_0/\partial S = 1/(S\sigma\sqrt{T})$, we have

$$\Delta = \Phi(x_1) + \frac{\varphi(x_1)}{\sigma\sqrt{T}} - \frac{Ke^{-rT}\varphi(x_0)}{S\sigma\sqrt{T}}$$

and then we can deduce that $\Delta = \Phi(x_1)$, as claimed above, from the identity

$$S\varphi(x_1) = Ke^{-rT}\varphi(x_0). \tag{1}$$

To see (1), note that

$$\frac{\varphi(x_0)}{\varphi(x_1)} = \frac{\exp(-x_0^2/2)}{\exp(-x_1^2/2)} = \exp((x_1^2 - x_0^2)/2) = \exp((x_1 + x_0)(x_1 - x_0)/2)$$

and

$$x_1 + x_0 = \frac{2(\log(S/K) + rT)}{\sigma\sqrt{T}}; \qquad x_1 - x_0 = \sigma\sqrt{T}.$$

Therefore

$$\frac{\varphi(x_0)}{\varphi(x_1)} = \exp\left(\log(S/K) + rT\right) = e^{rT}S/K,$$

and (1) follows.

The delta of a European put option. Recall the put-call parity relation

$$P = C + Ke^{-rT} - S$$

so differentiating with respect to S, and using the formula for Δ of the call option gives the delta of a put option:

$$\Delta_{put} = \Delta_{call} - 1$$

$$\Delta_{put} = \Phi\left(x_1\right) - 1$$

9.2 Delta hedging and delta neutrality

Delta hedging involves creating a position with $\Delta = 0$, i.e., a delta-neutral position. The aim is to eliminate uncertainty over the short-term change in the value of the portfolio. Since the delta of the underlying asset is 1, one way of doing this is to take a position of $-\Delta$ in the underlying asset. In fact, we already considered this in Section 8.5 where we argued that the value of a portfolio consisting of the derivative (value $f(t, S)$) together with $-\Delta$ units of the underlying, grows in a risk-free manner over a short period of time.

The delta neutral portfolio is held only over a very short period of time, since the delta of a derivative security changes over time. This means that the position in the underlying asset has to be frequently adjusted, i.e., rebalanced.

- The delta for a call option is always positive, it means that a long call can be hedged with a short position in the underlying, and a short call can be delta-hedged with a long position in the underlying.

- The delta for a put option is always negative, it means that a long put can be hedged with a long position in the underlying, and a short put can be hedged with a short position in the underlying.

A put option with a delta of -0.5 will drop by 0.5 in price for each unit rise in the price of the underlying, a call option with delta 0.5 will rise by 0.5 in price for each unit rise in the price of the underlying.

Consider a portfolio formed by m call options with price C and maturity T. If $m > 0$, then it means that we purchased the options (long position), if $m < 0$, then it means that we have sold the options (short position). We would like to delta-hedge these options. We add a quantity m_u of the underlying asset to the portfolio. We want the portfolio to be $\Delta-$neutral at time 0, i.e., the portfolio delta to be zero:

$$\Delta = 0 = m\Delta_c + m_u$$

where Δ_c is the call delta and we used the fact that delta of a unit of a stock is $1 (= \partial S / \partial S)$. Therefore we need to add

$$m_u = -m\Delta_c$$

of the underlying to the portfolio to make it delta-neutral.

If $\Delta = 0$, then

$$\delta f = \Theta \times \delta t + \mathcal{V} \times \delta\sigma + \frac{1}{2}\Gamma \times (\delta S)^2$$

The term with Δ is zero and drops out. If $\sigma = const$, then $\mathcal{V} \times \delta\sigma = 0$ and we arrive to

$$\delta f = \Theta \times \delta t + \frac{1}{2}\Gamma \times (\delta S)^2$$

If δS is small, then by properties of the log-normal model $(\delta S)^2 \approx \sigma^2 S^2 \delta t$, i.e., non-random. Over a very short period of time Θ and Γ are non-random. This means that there is no uncertainty in the change of the portfolio value, or, in other words, the portfolio is *risk-free*. But: it holds only for a *short period of time*. Delta-hedging only works for a short period of time during which the delta of the option is fixed. The hedge will have to be readjusted periodically to reflect changes in delta, which could be affected by changes in the share price, time to expiry, risk-free rate of return and volatility of the underlying.

Example. The stock price is $S = 100$ and the call option price is $C = 10$. Assume that the delta of the call is $\Delta_c = 0.5$. If we have sold the call option (option to buy 1 stock), then we can delta-hedge our position by buying 0.5 of stock. So, the portfolio delta is

$$\Delta = -0.5 + 0.5 = 0$$

The value of the portfolio is

$$f = -10 + 0.5 \times 100 = 40$$

If $S \to 100 + 0.2$, then we will get loss $-0.5 \times 0.2 = -0.1$ on the call option, but will gain 0.1 on the underlying stock. If $S \to 100 - 0.2$, then we will gain 0.1 on the call option, but will lose 0.1 on the underlying stock. The portfolio value remains unchanged in both cases.

In reality:

- the portfolio is not revised continuously, it can be revised only at discrete points of time;
- delta changes;
- the value of the risk-free investment changes (time value of money);
- there may be transaction costs.