6 The binary (binomial) pricing model.

We now describe a theory for arbitrage-free pricing of derivatives in a discrete-time world where in each time-step the stock price S changes in one of two possible ways.

6.1 The one-period binary model.

Suppose the time now is 0, and we have an option or other derivative with payoff $F(S(1))$ at time $T = 1$. The current stock price $S(0)$ is known and the stock price is assumed to change in one of just two possible ways; either $S(1) = uS(0)$ or $S(1) = dS(0)$, where $u > d > 0$. Let α denote the value at time 1 of a risk-free unit investment at time 0 ($\alpha = e^r$ in earlier notation).

We assume $u > \alpha > d$; otherwise there would be arbitrage opportunities.

Also, we assume $S(0)$ is known, and we can invest either in the stock or in a risk-free bank account.

The idea is that we look for an investment portfolio at time 0 which will generate a wealth at time 1 which precisely matches the payoff from the derivative. This is called a **replicating portfolio**. If we can find such an investment at time 0, we say that the inital value at time 0 of the investment portfolio is the **no-arbitrage price** of the derivative, since the derivative is sold for this price, the seller can invest the money received to be sure of meeting his obligation at time 1; if it is sold at some other price then there is an arbitrage opportunity for either the buyer or the seller.

To find the appropriate investment portfolio, suppose we invest cash amounts x in the stock and y in the bank account. Then the value of this portfolio at time 1 is

$$x(S(1)/S(0)) + y\alpha$$

and this will need to match the payoff from the derivative whether the stock goes to $S(0)u$ or $S(0)d$ if we choose x and y so that (with := denoting definition)

$$\begin{aligned} xu + y\alpha &= F(uS(0)) := z_u \\ xd + y\alpha &= F(dS(0)) := z_d \end{aligned}$$

This is two equations in two unknowns x and y, so can be solved. Subtracting the second equation from the first we get $x(u - d) = z_u - z_d$ so

$$x = (z_u - z_d)/(u - d), \tag{1}$$

and substituting this back in the first equation we get

$$\begin{aligned} \alpha(x + y) &= (\alpha - u)x + z_u \\ &= \frac{(\alpha - u)(z_u - z_d) + z_u(u - d)}{u - d} \\ &= \frac{z_u(\alpha - d) + z_d(u - \alpha)}{u - d} \\ &= z_u p + z_d(1 - p) \end{aligned}$$

where we set $p = \frac{\alpha - d}{u - d}$.

We can view this as the expected payoff, if we assume the probability of a 'good year' is chosen to be p. Note that under this probability assumption, the expected value of the stock at time 1 is

$$\begin{aligned} S(0)(up + d(1 - p)) &= S(0)\frac{u(\alpha - d) + d(u - \alpha)}{u - d} \\ &= \alpha S(0) \end{aligned}$$

so that the expected value of the stock at time 1 is the same as for an equal investment risk-free (this is called the principle of *risk-neutrality*). With this probability assumption, the no-arbitrage price of the derivative is

$$x + y = \alpha^{-1}(z_u p + z_d(1-p))$$

which is the expected discounted payoff
$E[\alpha^{-1}Z]$, under the risk-neutral probability assumption that p is chosen so that $E[S(1)] = \alpha S(0)$.

Finally, we have from (1) that if Δ denotes the amount of stock we buy as part of our replicating strategy, then

$$\Delta = \frac{x}{S(0)} = \frac{z_u - z_d}{S(0)u - S(0)d}.$$

That is, Δ is the (signed) difference between the payoffs associated with the higher and lower possible stock values at time 1, divided by the difference between the two stock values themselves.

Example. Suppose the current stock price is $S(0) = 100$, and assume that in the next year the stock price will change either to 140 (with probability 0.8) or to 70 (with probability 0.2). Assume the risk-free rate of interest is 10% compounded annually. Find the no-arbitrage price C of a 1-year call option with strike price 118, and find a an investment of amount C which perfectly replicates the payoff from this option.

Solution. The two possible outcomes at time 1 are $S(1) = 140$ or $S(1) = 70$. Note that although we have given the 'actual' probabilities of these two outcomes, this is *irrelevant* to the solution (although the actual two possible values of $S(1)$ *are* relevant) since we shall choose a different set of probabilities based on the principle of risk-neutrality. According to this principle, we choose $p = P[S(1) = 140]$ so that the expected discounted value of $S(1)$ is equal to $S(0)$, i.e. so that

$$\begin{aligned}(1.1)^{-1}E[S(1)] &= S(0), \text{ so} \\ E[S(1)] &= 1.1S(0) = 110\end{aligned}$$

and hence

$$\begin{aligned}140p + 70(1-p) &= 110 \text{ so} \\ 70p = 110 - 70 &= 40, \quad p = 4/7.\end{aligned}$$

The payoff Z is equal to $(S(1) - 118)^+$. Hence $Z = 22$ if $S(1) = 140$ and $Z = 0$ if $S(1) = 70$. With this risk-neutral choice of p, the expected discounted payoff is

$$\begin{aligned}(1.1)^{-1}E[Z] &= \frac{10}{11}(22p + 0(1-p)) \\ &= (\frac{10}{11}) \times 22 \times (\frac{4}{7}) = \frac{80}{7}.\end{aligned}$$

Hence, the no-arbitrage price for this option is $C = 80/7$.

We are asked also to find the investment strategy to replicate the payoff of the option. We have amount $C = 80/7$ to invest. We should buy Δ units of stock, where

$$\Delta = \frac{z_u - z_d}{S(0)u - s(0)d} = \frac{22 - 0}{140 - 70} = \frac{11}{35}.$$

Hence the amount of money we should invest in stock (x) is 100Δ which comes to $220/7$. The rest of our portfolio ($C - x$) we invest risk free, Here $C - x = -140/7 = -20$ so this should be our risk-free investment. In other words, borrow 20 at the risk-free rate, and combining this with the amount C we receive for the call, invest in 220/7 worth of stock. Together these give the total replicating portfolio, with net value C at time 0.

To check this really works, let us see what happens under the two possible scenarios. If the stock goes down the value at time 1 of the portfolio is

$$\frac{220}{7}\left(\frac{70}{100}\right) + (-20)(1.1) = 0$$

and if the stock goes up, the value of the portfolio at time 1 is

$$\begin{aligned} &\frac{220}{7} \times \frac{140}{100} - 20 \times \frac{11}{10} \\ &= 44 - 22 = 22 \end{aligned}$$

which matches the payoff in each case.

6.2 The two-period binary model.

For the multi-period binary model, we assume that we are given constants $d < \alpha < u$, such that the *proportionate* change in the stock in each time-period is either $u - 1$ or $d - 1$, i.e. $S(n+1)/S(n) = u$ or $S(n+1)/S(n) = d$ depending on whether year $n+1$ is a 'good year' or a 'bad year'. As before, α denotes the value after 1 time-unit of a unit risk-free investment.

Thus in the two-period model, we assume $S(1)/S(0)$ is either u or d, and $S(2)/S(1)$ is either u or d. Hence, the value of the stock at time 2 is $S(2)$ and $S(2)/S(0)$ can be either uu or ud or dd. We wish to price a derivative with payoff $F(S(2))$ at time 2. There are now three possible payoffs,

$$z_2 = F(S(0)uu), \quad z_1 = F(S(0)ud), \quad z_0 = F(S(0)dd).$$

We look for a **hedging strategy** (replicating portfolio), that is, a self-financing portfolio of stock and money in the bank, whose value at time 2 matches the payoff of the derivative.

We also assume we can *reinvest* our portfolio at time 1. That is, we make an initial investment in stocks and in the bank account at time 0. At time 1, having observed the stock price $S(1)$, we are allowed to transfer funds in our portfolio between the stocks and the bank account. However we do not introduce any 'new money' or remove any money at time 1; that is what 'self-financing' means.

Assuming we find such a hedging strategy, the initial value at time 0 of the strategy is a suitable price for the derivative, since otherwise there will be an arbitrage opportunity for the buyer or seller.

Let $p = \frac{\alpha-d}{u-d}$ and set $q = 1 - p$. As we have seen, this p has the property that if we assume $P[S(1)/S(0) = u] = p$ and $P[S(1)/S(0) = d] = q$ then $E[S(1)/S(0)] = \alpha$.

By the 1-step theory already considered, at time 1 we need to have wealth

$$\begin{aligned} z_u &= \alpha^{-1}(pz_2 + qz_1) \quad \text{if} \quad S(1) = uS(0) \\ z_d &= \alpha^{-1}(pz_1 + qz_0) \quad \text{if} \quad S(1) = dS(0) \end{aligned}$$

and hence by the 1-step theory (again!) the amount we need to invest at time 0 is

$$\begin{aligned} &\alpha^{-1}(pz_u + qz_d) \\ &= \alpha^{-2}(p(pz_2 + qz_1) + q(pz_1 + qz_0)) \\ &= \alpha^{-2}(p^2 z_2 + 2pqz_1 + q^2 z_0) \end{aligned}$$

which is α^{-2} times the expected payoff at time 2 if we assume that at each time-step the stock has probability p of having a 'good' year and q of having a 'bad' year, and different years are independent.

This quantity is the no-arbitrage price of the derivative. Let us denote this by C.

To find the first step of the hedging strategy, we find the value of the derivative at time 1, under the two possible scenarios for the value of $S(1)$. Let $f_1(x)$ denote the value of the derivative at time 1 if $S(1) = x$. Then set

$$\Delta = \frac{z_u - z_d}{S(0)u - S(0)d} = \frac{f_1(S(0)u) - f_1(S(0)d)}{S(0)u - S(0)d}$$

and at time 0, buy Δ units of stock, i.e. invest $S(0)\Delta$ in stock, and invest $C - S(0)\Delta$ risk free.

This is just the first stage of the replicating portfolio. We will need to readjust the investment at time 1, according to how the stock did in the first time step.

Example. Suppose that $S(0) = 100$, and $u = 1.4$ and $d = 0.7$ (as before), but now suppose $\alpha = 1$. Find the no-arbitrage price for a put option with strike 140 and maturity date 2. Also find describe the initial investment at time 0 for a hedging strategy for this option

Answer: We need to choose p so that

$$E[S(1)/S(0)] = \alpha = 1,$$

and in this case this will be $p = 3/7$. The payoff is $(140 - S(2))^+$, and the possible values for $S(2)$ and associated probabilities (under risk-neutrality) are

- $100(1.4)^2 = 196$ with probability $p^2 = 9/49$, giving a payoff of 0.
- $100(1.4)(0.7) = 98$ with probability $2p(1-p) = 24/49$, giving payoff 42.
- $100(0.7)^2 = 49$ with probability $(1-p)^2 = 16/49$, giving payoff 91.

Hence if the payoff is denoted Z, since $\alpha = 1$ the no-arbitrage price is the expected discounted payment which is

$$\begin{aligned}\alpha^{-2}E[Z] &= (42 \times \frac{24}{49}) + (91 \times \frac{16}{49}) \\ &+0 \times \frac{9}{49} = \frac{352}{7} = 50.2 = P.\end{aligned}$$

To find the first step of the hedging strategy, note that after 1 year, we need our wealth in the portfolio to be enough to match the possible payoffs one year later. One possibility is that the first year is a 'good' year and $S(1) = 140$. In this case the value of $S(2)$ will be either 196 or 98, so under risk-neutrality the expected payoff, if we know $S(1) = 140$, is

$$z_u = 0 \times (\frac{3}{7}) + 42 \times (\frac{4}{7}) = 24$$

and therefore, our investment needs to be worth 24 at time 1 if the first year is good.

If the first year is bad, then the expected payoff for the second year is

$$z_d = (42 \times \frac{3}{7}) + (91 \times \frac{4}{7}) = 70$$

Hence,

$$\Delta = \frac{z_u - z_d}{S(0)u - S(0)d} = \frac{24-70}{140-70} = -\frac{23}{35}$$

and the amount we initially invest in stock should be $\Delta S(0)$ which comes to $-(23/35)100$ which is $-460/7$. The amount we initially invest risk-free is $P - \Delta S(0)$, which is

$$\frac{352}{7} - (-\frac{460}{7}) = \frac{812}{7} = 116.$$

Together, these invstments form the first step of our replicating portfoliao. Investing in a negative amount of stock means short selling, so at time 0 the strategy is to short sell $460/7 = 65.7$ worth of stock, and combine this with the initial funds of $352/7$ for the portfolio, to invest 116 in the risk-free bank account.

Let us check this works. The value of the portfolio at time 1, in the case of a good year in year 1, is

$$1.4(\frac{-460}{7}) + 116 = 24$$

If the first year is bad, i.e. if $S(1) = 70$, the value of the portfolio is

$$116 - 0.7(\frac{460}{7}) = 70$$

so in each case the wealth at time 1 from our investment is indeed enough to reinvest at time 1 to match the payoff at time 2, whether the first year is good or bad.

6.3 The n-step binary model Now consider the 3-step model. For the three-period binary model, we assume that $S(1)/S(0)$ is either u or d, and $S(2)/S(1)$ is either u or d and $S(3)/S(2)$ is either u or d. Hence, the value of the stock at time 3 is $S(3)$ and $S(3)/S(0)$ can be either uuu or uud or udd or ddd. We wish to price a derivative with payoff $F(S(3))$ at time 3. There are now four possible payoffs,

$$\begin{aligned} z_3 = F(S(0)uuu), &\quad z_2 = F(S(0)uud), \\ z_1 = F(S(0)udd), &\quad z_0 = F(S(0)ddd). \end{aligned}$$

Again we look for a hedging strategy, that is, a self-financing portfolio of stock and money in the bank, whose value at time 3 matches the payoff of the derivative. We now assume we can reinvest our portfolio at time 1 and again at time 2. That is, we make an initial investment in stocks and in the bank account at time 0. At time t ($t = 1$ or $t = 2$), having observed the stock price $S(t)$, we are allowed to transfer funds in our portfolio between the stocks and the bank account, but without introducing any 'new money' or removing any money at that time.

By the earlier results, at time 1 we need our wealth to be

$$\begin{aligned} &\alpha^{-2}(p^2 z_3 + 2pqz_2 + q^2 z_1) \quad \text{if} \quad S(1) = uS(0) \\ &\alpha^{-2}(p^2 z_2 + 2pqz_1 + q^2 z_0) \quad \text{if} \quad S(1) = dS(0) \end{aligned}$$

and therefore the wealth at time 0 needs to be

$$\begin{aligned} &\alpha^{-3}[p(p^2 z_3 + 2pqz_2 + q^2 z_1) \\ &\quad +q(p^2 z_2 + 2pqz_1 + q^2 z_0)] \\ &= \alpha^{-3}(p^3 z_3 + 3p^2 q z_2 + 3pq^2 z_1 + q^3 z_0). \end{aligned}$$

In general, using proof by induction it can be shown that for the n-period binary model, the no-arbitrage price for a derivative paying amount z_k if there are k good years and $n-k$ bad years, is given by

$$\alpha^{-n} \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} z_k$$

which is the same as the expected discounted payoff under the risk-neutral assumption that the number of good years is binomially distributed with parameter p, so that the expected value of S_{i+1}/S_i is equal to α each year and successive years are independent.

In other words, the no-arbitrage price is given by $\alpha^{-n}E[Z]$, where $Z = F(S(n))$, and we make the risk-neutral assumptions described above.

The same principle applies for more complicated derivatives such as barrier options for which the payoff depends not only on the stock price at time n but on the sequence of stock prices $S(0), S(1), \ldots, S(n)$. Again, the no-arbitrage price equals the expected discounted payoff under the assumption of risk-neutrality.

Example. Suppose the current stock price is 100, and each year the stock price goes either up or down by 10 per cent, with equal probability. Suppose the risk-free rate of interest is 5 per cent (compounded annually).

What is the no-arbitrage price for a 4-year binary call option with strike price 110? What about an up-and-out 4-year call option with strike price 100 and barrier 120?

Answer. The 'equal probability' part of the question is irrelevant, we have to re-set the probability of the stock going up to p each year, with p chosen so that $E[S(1)/S(0)] = 1.05$. The correct choice of p is $p = 3/4$. Then the number of 'good' years in the next four years, N is binomial with $n = 4, p = 3/4$. The stock price in 4 years time is

$$N = 4 \Rightarrow S(4) = 100(1.1)^4 = 146.41$$
$$N = 3 \Rightarrow S(4) = 100(1.1)^3(0.9) = 119.79$$
$$N = 2 \Rightarrow S(4) = 100(1.1)^2(0.9)^2 = 98.01$$

and so on. The binary call pays off 1, provided $S(4) \geq 110$, i.e. provided $N \geq 3$. Otherwise the payoff is zero. Hence, the expected payoff for the binary call is

$$\begin{aligned} 1 \times P[N \geq 3] + 0 \times P[N < 3] \\ = p^4 + 4p^3(1-p) \\ = 0.3164 + 0.4219 = 0.7383 \end{aligned}$$

and the no-arbitrage price is $(1.05)^{-4}(0.7383)$ which comes to 0.6074.

For the up-and-out option, there is no payoff unless $N = 3$. Also we need the first two years to not both be good, otherwise we go above the barrier at time 2. So the two ways we can end up with a non-zero payoff (of 19.79) is if the sequence of years is *uduu* or *duuu*. Each of these has probability $p^3(1-p)$ which comes to 0.1055 and so the no arbitrage price, i.e. the expected discounted payoff, is

$$2 \times (0.1055) \times 19.79 \times (1.05)^{-4} = 3.44$$

To sum up: in the binary model, to find the no-arbitrage price, we IGNORE the 'actual' values of P[up] and P[down], and instead compute the expected discounted payoff, assuming the **risk-neutral** value of P[up], i.e. the value of P[up] which makes the expected growth the same for a unit investment in stock as for a unit investment risk-free.