5.4 Expectation.

The **expected value** or **expectation** of a discrete random quantity X is a number $E[X]$ given by the formula

$$E[X] = \sum_x xP[X = x]$$

For example, if X is binomial with parameters $n = 2$ and $p = 1/2$, then

$$E[X] = (0 \times 1/4) + (1 \times 1/2) + (2 \times 1/4) = 1$$

and if X is binomial with parmeters $n = 3$ and $p = 1/4$ then

$$\begin{aligned} E[X] &= (0 \times 27/64) + (1 \times 27/64) \\ &\quad + (2 \times 9/64) + (3 \times 1/64) \\ &= 0 + 27/64 + 18/64 + 3/64 \\ &= 48/64 = 3/4 \end{aligned}$$

It can be shown in general that if X is binomial with parameters n and p (so X can be thought of as representing the number of successes in n independent trials each with probability p of success) then

$$E[X] = np.$$

In general, if X is a random quantity representing a numerical score arising from a repeatable experiment, then the number $E[X]$ represents the 'average score' arising from a large number of repetitions of the experiment.

Here are some general properties of expectations.

- If a is a number and X is a random quantity, then if $P[X \geq a] = 1$, then $E[X] \geq a$.
- If X is a random quantity and a is a constant then $E[aX] = aE[X]$.
- If X is a discrete random quantity and $g(x)$ is a real-valued function defined for all possible values x of X then the random quantity $g(X)$ has expected value

  $$E[g(X)] = \sum_x g(x)P[X = x]$$

  where the sum is over all possible values x for X. For example, if X is the score on a fair die,

  $$E[(X - 3)^2] = \sum_{x=1}^{6} (x - 3)^2 \times \frac{1}{6} = \frac{19}{6}$$

- If X and Y are two random variables on the same sample space, then it is always the case that

  $$E[X + Y] = E[X] + E[Y]$$

We say X and Y are **independent** (discrete) random variables if for any two numbers a, b we have

$$P[(X=a)\cap(Y=b)] = P[X=a]P[Y=b]$$

where $P[X=a]$ means $P[\{\omega : X(\omega)=a\}]$ and $P[(X=a)\cap(Y=b)]$ means

$$P[\{\omega : X(\omega)=a \text{ and } Y(\omega)=b\}].$$

Similarly, we say discrete random variables X, Y, Z on the same sample space are **mutually independent** if for all numbers a, b, c,

$$\begin{aligned} &P[(X=a)\cap(Y=b)\cap(Z=c)] \\ &\quad = P[X=a]P[Y=b]P[Z=c] \end{aligned}$$

and likewise for more than three random variables.

It is a **fact** that if X and Y are independent random variables, then

$$E[XY] = E[X]E[Y]$$

If X, Y, Z are mutually independent random variables then

$$E[XYZ] = E[X]E[Y]E[Z].$$

Example. Suppose that the price of a stock is currently 100, and each month it is assumed that the stock price either increases by 50% (with probability 1/4) or goes down by 33.333% (with probability 3/4), independently of other months.

(1) What is the probability distribution of $S(3)$, the stock price in 3 months' time?

Answer. Let N be the number of 'good months' in the next three months, i.e. the number of times the stock goes up in the next 3 months. Then N is binomial with $n=3, p=1/4$, For $k=0,1,2,3$, if $N=k$ we have

$$S(3) = 100(3/2)^k(2/3)^{3-k}.$$

In other words,

$$S(3) = 100(3/2)^N(2/3)^{3-N}.$$

The possible values for $S(3)$ are $100(3/2)^3 = 337.5$, or $100(3/2)^2(2/3) = 100(3/2) = 150$, or $100(3/2)(2/3)^2 = 100(2/3) = 66.67$, or $100(2/3)^3 = 29.630$. Hence, the probability distribution of $S(3)$ is given by

$$\begin{aligned} P[S(3)=337.5] &= P[N=3] = 1/64 \\ P[S(3)=150] &= P[N=2] = 9/64 \\ P[S(3)=66.67] &= P[N=1] = 27/64 \\ P[S(3)=29.630] &= P[N=0] = 27/64 \end{aligned}$$

(2) What is the expected value of the payoff for a call option with maturity in 3 months' time, and strike price of 100, and hence with payoff of $(S(3)-100)^+$?

Answer. The payoff X satisfies $P[X = 237.5] = 1/64$ and $P[X = 50] = 9/64$ and $P[X = 0] = 54/64$ so that the expected payoff is

$$E[X] = (237.5 \times \frac{1}{64}) + (50 \times \frac{9}{64})$$

(3) What is the expected value of the payoff for a straddle consisting of a put option and a call option both with maturity in 3 months' time, and strike price of 100?

Answer. Let Y denote the payoff for the put option, so that $Y = (100 - S(3))^+$. Then $P[Y = 0] = 1/64 + 9/64$, and $P[Y = 33.33] = 27/64$, and $P[Y = 70.37] = 27/64$. Hence

$$E[Y] = (70.37 \times \frac{27}{64}) + (33.33 \times \frac{27}{64}) = 43.75$$

The total payoff is Z for the straddle is $Z = X + Y$, so the expected payoff for the straddle is

$$E[Z] = E[X] + E[Y]$$

(4) What is the expected value of the stock price after 3 months?

Answer: there are at least two ways to do this. The direct approach is to say that

$$E[S(3)] = \sum_x xP[S(3) = x]$$

Use the probability distribution of $S(3)$ to say that

$$\begin{aligned} E[S(3)] &= \left(337.5 \times \frac{1}{64}\right) + \left(150 \times \frac{9}{64}\right) \\ + \left(66.67 \times \frac{27}{64}\right) &+ \left(29.63 \times \frac{27}{64}\right) = 66.99 \end{aligned}$$

Another way to do this would be to note that $S(3) = 100D_1D_2D_3$, where D_i is the proportionate change in the ith month, so

$$P[D_i = 3/2] = 1/4, \quad P[D_i = 2/3] = 3/4$$

$$\begin{aligned} E[D_i] &= \frac{3}{2} \times \frac{1}{4} + \frac{2}{3} \times \frac{3}{4} \\ &= \frac{3}{8} + \frac{1}{2} = \frac{7}{8} \end{aligned}$$

and D_1, D_2, D_3 are independent so

$$\begin{aligned} E[S(3)] &= E[100D_1D_2D_3] \\ &= 100E[D_1]E[D_2]E[D_3] \\ &= 100\left(\frac{7}{8}\right)^3 = 66.99 \end{aligned}$$

which agrees with the value of $E[S(3)]$ obtained by the first method.