4 Exotic options.

A *derivative security* (or *derivative* for short) is a financial instrument whose payoff is *derived* from an asset price (i.e., determined by the asset price) at a future date. The basic call or put options which we have already described are also called 'vanilla' options. These are the most important types of options, but there are many other types of derivatives besides vanilla options. These are sometimes called 'exotic options' and we describe some of them options in this section. In some cases (but not all), these derivatives can be obtained by combining positions in vanilla options.

4.1 Spreads.

A spread is a portfolio combining long and short positions in two (European) options on the same asset/stock with the same maturity date T but with different strike prices. (taking the long position in an option means you buy the option, and taking the short position means you sell it). There are **bull spreads** and **bear spreads**; a *bull* is a person who believes the stock price will rise, while a *bear* is someone who believes it will fall; a bull spread is a spread that a bull would be interested in owning, while a bear spread is a spread that a bear would be interested in owning.

Terminology: If $f(x)$ is a numerical-valued function defined for all (real) numbers x, we say the function f is an *increasing* function if $f(x) \leq f(y)$ whenever $x < y$ (i.e. if the graph of f 'slopes up'), and that f is a *decreasing* function if $f(x) \geq f(y)$ whenever $x < y$ (i.e., if the graph of f 'slopes down').

We note here that the price of a call option is a decreasing function of the strike price. That is, if two otherwise identical calls have strike price K_1 and K_2 respectively, with $K_1 < K_2$, then the price of the first option is more than that of the second option, since the payoff $(S(T) - K_1)^+$ from the first option is at least as big as the payoff $(S(T) - K_2)^+$ from the second option.

For a **bull spread** you take a long position on a call option with strike K_1 and a short position in a call option with strike K_2, with $K_2 > K_1$ (and with both options being on the same asset with the same maturity date T). Then the payoff at time T is a function $f(S(T))$ with f given by

$$f(x) = (x - K_1)^+ - (x - K_2)^+$$

so that

$$f(x) = \begin{cases} 0 & \text{if} \quad x < K_1 \\ (x - K_1) & \text{if} \quad K_1 \leq x < K_2 \\ (K_2 - K_1) & \text{if} \quad x \geq K_2 \end{cases}$$

Note that this payoff is nonnegative, so you can expect to pay a positive amount at time 0 to obtain a bull spread. To see this another way, note that the price of an option is a decreasing function of strike price so you pay more for the long position in the option with strike price K_1 than you receive for the short position in the option with strike K_2.

Also, the payoff is an increasing function of $S(T)$ and thus holding this spread would appeal to a bull. In effect, holding a spread is like holding a call option but with a cap on the maximum profit that can be made.

A **bear spread** is a combination of a long position in a put option with strike K_2 with a short position in a put option on the same share with strike K_1 (again with $K_1 < K_2$), both with the same maturity date. The payoff at time T is a function $g(S(T))$ with g given by

$$g(x) = (K_2 - x)^+ - (K_1 - x)^+$$

so that

$$g(x) = \begin{cases} K_2 - K_1 & \text{if} \quad x < K_1 \\ K_2 - x & \text{if} \quad K_1 \leq x < K_2 \\ 0 & \text{if} \quad x \geq K_2 \end{cases}$$

This time the payoff is a decreasing function of $S(T)$ so holding this spread might appeal to a bear.

One may also obtain a bull spread using puts or a bear spread using calls, but in both cases one would need to add cash to the portfolio to make the payoff nonnegative, as we now describe.

To get a bull spread using puts, take a long position in a put with strike K_1 and a short position in a put with strike K_2. The payoff net payoff from these puts will be $-g(S(T))$, and if one adds a risk-free investment of $(K_2 - K_1)e^{-rT}$ at time 0 to the portfolio, the total value at time T of the portfolio comes to $-g(S(T)) + (K_2 - K_1)$, which is the same as $f(S(T))$.

Similarly one could get a bear spread using calls, by taking a long position in a call with strike price K_2 and a short position in a call with strike price K_1, together with a cash amount $(K_2 - K_1)e^{-rT}$ invested risk-free at time zero. Then the total value of the portfolio at time T is $-f(S(T)) + (K_2 - K_1)$, which comes to $g(S(T))$.

For both bull and bear spreads (as described above), the payoff ($f(S(T))$ or $g(S(T))$) is nonnegative and could be strictly positive, so we would expect to have to pay an amount up-front at time 0 to achieve the bull or bear spread portfolios described above. This up-front payment has not been included in the description of payoffs above.

4.2 Straddles and strangles

A **straddle** obtained by buying both a call option and and put option in the same asset with the same strike price K and the same maturity date T. The payoff is given by $f(S(T))$ with

$$f(x) = (x - K)^+ + (K - x)^+ = |x - K|.$$

A **strangle** is obtained by buying both a call option with strike price K_2 and a put option with strike price K_1, with $K_1 < K_2$. The payoff for the strangle is $g(S(T))$ with the function g given by

$$\begin{aligned} g(x) &= (x - K_2)^+ + (K_1 - x)^+ \\ &= \begin{cases} (K_1 - x) & \text{if} \quad 0 \leq x < K_1 \\ 0 & \text{if} \quad K_1 \leq x < K_2 \\ (x - K_2) & \text{if} \quad x \geq K_2 \end{cases} \end{aligned}$$

Ownership of either a straddle or a strangle may be appealing to someone who believes the stock price is going to change dramatically but does not know whether the change will be up or down. This situation could arise, for example, if a court ruling is impending that will significantly affect the value of the stock, in a positive or negative way according to the ruling.

4.3 Binary options.

In a binary option, the payoff is 1 or zero. For a binary call, the payoff at the expiry time T is 1 if the asset price at time T is at least the strike price K and is zero otherwise. For a binary put the payoff at time T is 1 if the asset price at time T is below K and is zero otherwise.

There is a nice put-call parity relation for these options. Let P_1 and C_1 denote the price of a binary put and a binary call respectively, with the same strike price and the same maturity date T. Then

$$P_1 + C_1 = e^{-rT}$$

This is because if you hold both the binary call and the binary put simultaneously, then precisely one of them will pay off a unit amount at time T, whatever the stock price does.

We note that the binary call (with strike K) can be approximated by a bull spread in n units of the asset with strike prices $K_1 = K - (1/n)$ and $K_2 = K$, with n taken to be very large. Similarly, the binary put with strike K can be approximated by a bear spread in n units of the asset with strike prices $K_1 = K - (1/n)$ and $K_2 = K$, with n taken to be very large.

4.4 Options with path-dependent payoff.

We now consider some exotic options where the payoff depends not only on the value of the stock at the maturity date T but on the behaviour (path) of the the stock price through the whole time-period from 0 to T (in some sense this is true for American options, but the options we describe now are of European type in the sense that the holder of the option does not have any decision to make before time T). The options we describe are **barrier options**, **lookback options** and **Asian options**.

For a barrier option, the payoff depends on whether or not the stock price crosses some specified level (barrier) H before time T. If it is a *knockout* option then the option becomes worthless if the barrier is crossed at any time before time T, and the payoff will be the same as for a regular option if the barrier is not crossed. If $H > S(0)$ it is an *up-and-out* option, and if $H < S(0)$ it is a *down-and-out* option. For example, if H exceeds the current stock price $S(0)$ and also exceeds the strike price K, we could have an *up-and-out* European call option. The payoff would be as for a normal call option unless the stock price gets as high as H, in which case the payoff is zero. In other words, for this option, setting

$$M_T = \max\{S(t) : 0 \leq t \leq T\}$$

the payoff is

$$\begin{cases} 0 & \text{if} \quad S(T) < K \\ S(T) - K & \text{if} \quad S(T) \geq K, M_T \leq H \\ 0 & \text{if} \quad M_T > H \end{cases}$$

Similarly, for a down-and-out European put option with barrier at $H < S(0)$ (and $H < K$), if we set

$$m_T = \min\{S(t) : 0 \le t \le T\},$$

then the payoff is

$$\begin{cases} 0 & \text{if} \quad S(T) \ge K \\ K - S(T) & \text{if} \quad S(T) < K, m_T > H \\ 0 & \text{if} \quad S(T) < K, m_T \le H \end{cases}$$

Similarly, there are *knock-in* (either *up-and-in* or *down-and-in*) barrier options where the payoff is zero unless the stock price crosses a barrier before time T, and if it does so the payoff is the same as for a regular option.

For a **lookback** option, the strike price is not fixed at time 0 but is found at time T by "looking back" over the lifetime of the option (i.e., over the time-period from 0 to T) and setting the strike price to be m_T (the minimum stock price over that period) in the case of a lookback call, or to be M_T (the maximum stock price over that period) in the case of a lookback put. Hence the payoff at time T is

$$\begin{aligned} S(T) - m_T \quad & \text{for a lookback call} \\ M_T - S(T) \quad & \text{for a lookback put.} \end{aligned}$$

For an **Asian** option, the payoff is determined not by the asset price at expiry $S(T)$, but by the *average* price of the asset over some specified range of times (e.g., over the entire range of times from 0 to T). If we denote this average price by S_{Av}, the Asian option payoff at time T is

$$\begin{aligned} (S_{\text{Av}} - K)^+ \quad & \text{for an Asian call} \\ (K - S_{\text{Av}})^+ \quad & \text{for an Asian put} \end{aligned}$$