3 Options

3.1 Types of options

Options are available in stocks, in indices, in foreign currencies, in commodities, in interest rates and so on. Understanding their pricing is important because they are so widely available, and also because they can be combined to create more sophisticated financial instruments.

As mentioned earlier, a call option is the option to buy an asset for a specified price K, while a put option is the option to sell. For a European option, the decision on whether to exercise an option is made at the expiry date T, whereas an American option can be exercised at any time before T.

The payoff at time T from a European call option is $(S(T)-K)^+$ and from a European put option is $(K-S(T))^+$. In the case of American options, the payoff takes place at the moment of exercise t, where $t \leq T$ and we set $t = T$ if the option is not exercised. For American options, the payoff is $(S(t)-K)^+$ for a call option and $(K-S(t))^+$ for a put.

Options provide an alternative to futures/forwards contracts as a means of **hedging** against exposure to future risks. Consider, for example, a jewelry merchant needing gold for producing jewelry, who wants to hedge against uncertainty about the future price of gold (needed to be able to make jewelry). One method of hedging would be to take a long position in a gold futures contract. Another would be to acquire a call option in gold. Then he is guaranteed not to have to pay more than the strike price for the gold.

Note that in the above example, the jewelry merchant will not have to make an upfront payment if he enters into a futures contract so as to hedge his risk. If he acquires an option, however, he will have to make an upfront payment for the option. This payment is called the price, or *premium*, of the option.

To *write* an option is to sell an option. The writer of a call option is obliged to deliver the underlying asset to the option holder (in exchange for the strike price) if the call is exercised. The writer of a put option is obliged to buy the underlying asset from the option holder if the put is exercised. The writer keeps the premium whether or not the option is exercised.

At a given time t with $0 \leq t \leq T$, we say an option is *in the money* if exercising the option right now (i.e. at time t) (where in the case of a European option we pretend this is allowed) would make us a profit. So a call option is in the money if $S(t) > K$, and a put option is in the money if $S(t) < K$. Similarly, we say an option is *out of the money* if exercising it right now would lose you money (eg if $S(t) < K$ for a call option) and that it is *at the money* if you would break even by exercising it right now, i.e. if $S(t) = K$.

The *intrinsic value* at time t of an option is the maximum of the profit you'd make if you exercise it right now (again pretending this is allowed in the European case), and zero. So the intrinsic value is $(S(t)-K)^+$ for a call and $(K-S(t))^+$ for a put.

The *time value* of an option is the amount by which the the option's value exceeds its intrinsic value. That is, we have

$$\text{Option value} \;=\; \text{intrinsic value} \;+\; \text{time value}$$

For an American option the value of the option at time t is at least as much as the intrinsic value because you can exercise right now, but you might be better off waiting, so the time value is non-negative. For European options the time value could conceivably be negative.

If t is close to T we'd expect the option value to be close to the intrinsic value and the time value to be close to zero.

For example, if a 3-month put option on a certain stock is trading at £3, when the current stock price is £30 and the strike price is £31, then the intrinsic value of the option is £1 and the time value is £2.

3.2 Factors affecting option prices.

In this course we shall discuss the pricing of options at some length. We can expect the following factors to affect the price of a stock option:

1. *The current stock price.* Call options become more valuable if the stock price increases; put options become more valuable if the stock price decreases.

2. *The strike price.* As the strike price (K) is raised, the price of a call option (C) goes down and the price of a put option (P) goes up. For example, for options in Barclay's we might have

K	580	600
C	34.25	22.25
P	13.50	21.75

3. *The time to expiration.* Prices of both put and call American options increase as the option maturity date increases. because the owner of the long-life option has all exercise opportunities open to the owner of the short-life option - and more. For example, for Royal Bank of Scotland share options, could have:

	Oct	Nov	Dec
C(K=500)	26.5	34.00	42.00
P(K=500)	14.00	19.5	24.5

   European put and call options do not necessarily become more valuable as the time to expiration increases, but often do so, because a long-maturity call option has longer for the price of the option to change dramatically, and so a better chance of a large payoff. However, this effect is offset by the time value of money.

4. *The risk-free interest rate* (r). This affects the time-value of the future payoff from the option, so we expect changes in the interest rate to have an effect on the option's value.

5. The *dividends* expected during the life of the option. We have seen that these affect the pricing of futures, and similarly we might expect them to affect the pricing of options.

6. The *volatility* of the stock price. This is a measure of the 'amount of randomness per unit time' in the fluctuations of the stock price (we shall be more precise about this later). An option on a more volatile stock tends to be more valuable than on a less volatile stock because the stock is more likely to take a value dramatically above (or below) the strike price leading to a big payoff for a call (put) option.

3.3 Bounds on option prices.

Let us write C for the price (at time 0) of a call option and P for the price of a put option, on a stock (or other asset) with value $S(t)$ at time t. Let T denote the expiry date and K the strike price.

An *upper bound* on an unknown quantity is a number that is definitely bigger than or equal to than that quantity. A *lower bound* on an unknown quantity is a number that is definitely smaller than or equal to that quantity. We can obtain the following upper and lower bounds on option prices by arbitrage arguments,

First we give some upper bounds. For either a European or American call option, assuming the underlying asset has no storage costs, we have

$$C \leq S(0)$$

To see this note that it is not as good to hold a call option as to hold the stock itself. If the inequality was violated (i.e. if we had $C > S(0)$) we could sell a call option for amount C, buy a unit of stock for amount $S(0)$ and hold it until the option was exercised, at which point we'd use the stock we hold to meet our obligation under the option. This would ensure a profit.

Next, note that for a European put option we have

$$P \leq Ke^{-rT}$$

since the payoff at time T is $(K - S(T))^+ \leq K$, and this is less than the payoff from a risk-free investment of Ke^{-rT} at time 0 (the arbitrage strategy if $P > Ke^{-rT}$ is to sell the put option for amount P and invest Ke^{-rT} to use if the option is exercised at time T). In the case of an American put option, the payoff could happen at any time after time 0, and the best we can say for sure (assuming $r \geq 0$) is that $P \leq K$.

Next we give a lower bound for the price of the call option. In the case of the European option, assuming the stock does not pay any dividends, the price C satisfies

$$C \geq S(0) - Ke^{-rT}.$$

To see this consider a portfolio of 1 option and Ke^{-rT} in cash. The value of this portfolio is $C + Ke^{-rT}$ at time zero, and if the option is held to time T while the cash invested at the risk-free rate, then at time T the accumulated cash of K will be enough to exercise the option (if one wishes to) to end up with amount $S(T)$, the same as if one initially held amount $S(0)$ and invested in stock. Hence the initial value of the portfolio is at least $S(0)$, i.e. $C + Ke^{-rT} \geq S(0)$.

To put this another way, if the above inequality failed and we had $C + Ke^{-rT} < S(0)$. you could short sell the share, promising to return it at time T, use the money thus raised to buy the option for C and to invest Ke^{-rT} risk free, keeping the rest as profit, and then exercise the option using the cash investment at time T to return the stock.

Thus $S(0) - Ke^{-rT}$ is a lower bound for the price of a European call option, and hence also for the price of an American call option, since an American call is worth at least as much as a European call.

A lower bound for the price of a European (or American) put option is

$$P \geq Ke^{-rT} - S(0).$$

To see this consider a portfolio of one put option plus one unit of stock. If the stock in the portfolio is retained to time T and then the option is exercised, you end with amount K at time T (with no risk), so the initial value $P + S(0)$ of the portfolio is at least as big as the initial value Ke^{-rT} of a risk-free investment worth K at time T. Hence $P + S(0) \geq Ke^{-rT}$.

In terms of arbitrage: if $P + S(0) < Ke^{-rT}$ we could borrow Ke^{-rT}, buy the portfolio just described (and keep the extra as profit), then at time T exercise the option and repay the loan.

The previous inequalities need modifying if the stock pays **dividends**. Suppose the present value of the dividends paid by the stock between time 0 and time T is known to be D. Then the upper and lower bounds for European options become

$$\begin{aligned} C &\leq S(0) - D \\ P &\leq Ke^{-rT} \\ C &\geq S(0) - D - Ke^{-rT} \\ P &\geq D + Ke^{-rT} - S(0) \end{aligned}$$

The arguments to derive these are similar to before, using the fact that an initial amount of $S(0) - D$ accumulates in value to $S(T)$ by time T if we use it to buy the stock and borrow amount D, using the dividend from the stock to repay the cash borrowed.

Let us go through just one of the arbitrage arguments: suppose the third inequality above failed and $C < S(0) - D - Ke^{-rT}$. Then an arbitrageur could short sell the stock, buy the call, invest amount D risk-free and use this to match the dividend payments on the stock, invest Ke^{-rT} risk-free and use this to exercise the call at time T.

3.4 American call options on non-dividend paying stock.

For a non-dividend paying stock, it is NEVER optimal to exercise an American call option early. Indeed, we have seen above that for such an option the option price C at time zero satisfies

$$C \geq S(0) - Ke^{-rT}$$

and by a similar argument, at any intermediate time t $(0 < t < T)$, the option price $C(t)$ satisfies

$$C(t) \geq S(t) - Ke^{-r(T-t)}$$

and hence (assuming $r > 0$) we obtain $C(t) > S(t) - K$ (actually this holds even if $r = 0$ because the displayed inequality is actually strict, but we omit details here), and hence the value of the option exceeds the profit the option holder makes if they exercise the option. Hence the option holder should not exercise the option since they'd be better off selling it.

This justifies the above assertion, and shows that in fact for a call option on non-dividend paying stock, an American option is worth the same as the corresponding European option.

This argument does not apply for put options or for options in dividend-paying stocks. In these cases, it might be better in some circumstances to exercise the American option early, so in these cases the American option is worth more than the corresponding European option.

3.5 Put-call parity.

Consider the following two portfolios involving European options in a non-dividend-paying stock:

- **Portfolio A** (call plus cash): One call option plus a risk-free investment of Ke^{-rT} at time zero.
- **Portfolio B** (put plus stock): One put option plus one unit of stock.

By time T the value of Portfolio A is

$$\begin{aligned}(S(T)-K)^{+}+K &= \max(S(T)-K,0)+K\\ &= \max(S(T),K)\end{aligned}$$

while the value of Portfolio B is

$$\begin{aligned}(K-S(T))^{+}+S(T) &= \max(K-S(T),0)+S(T)\\ &= \max(K,S(T)).\end{aligned}$$

Hence, both portfolios have value $\max(K, S(T))$ at time T. Hence they must have the same value at time zero. Hence

$$C+Ke^{-rT}=P+S(0).$$

If this does not hold, then there would be arbitrage opportunities.

Example. A non dividend paying stock price is currently £31. A 3-month European call option with strike price £30 is trading for £3, while a European put with the same strike price and expiry date is trading for £2.25. The risk-free interest rate with continuous compounding is 10%. What arbitrage opportunity exists?

Answer. Here we have $S(0) = 31$, $K = 30$, $r = 0.1$, $T = 0.25$, $C = 3$ and $P = 2.25$, so

$$\begin{aligned}C+Ke^{-rT} &= 3+30e^{-0.025}=32.26;\\ P+S(0) &= 33.25\end{aligned}$$

so here $P + S(0) > C + Ke^{-rT}$ so the put price is too high.

The arbitrage involves selling portfolio B (Put plus cash) and buying portfolio A (call plus stock). That is, we short sell the stock and sell the put, receiving amount $P + S(0)$. Using some of this money, we buy the call and invest amount Ke^{-rT} risk-free, and keep the rest as profit (the profit is $33.25 - 32.26 = 0.99$). At time T, if $S(T) \geq K$ we exercise the call and end up with 1 unit of stock which we use to return the stock we borrowed when short selling at time 0 (and the put we sold is worthless). If $S(T) \leq K$, we do not exercise our call but the holder of the put should exercise; we use our accumulated investment of K to pay for the stock we have to buy as a result, and then hold a stock which we use to make up our deficit in stock.

Conversely if we had $P + S(0) < C + Ke^{-rT}$ we'd sell portfolio A and buy portfolio B. That is, we'd sell the call option and borrow amount Ke^{-rT} and use the resulting funds to buy the put option and the stock.

What if the stock provides a dividend yield?

Suppose the dividend yield accrues at rate q per unit of stock value. Consider the following portfolios with European options:

- **Portfolio A:** one call option plus Ke^{-rT} in cash.
- **Portfolio B:** one put option plus $S(0)e^{-qT}$ worth of shares.

In the first portfolio, suppose the cash is invested risk-free up to time T as before. Then the value of the first portfolio at time T is $\max(K, S(T))$ as before.

In the second portfolio suppose the initial $S(0)e^{-qT}$ worth of shares has all share income reinvested into the shares up to time T. Then as described in Section 2.5, the value of the shares by time T will be $S(T)$ so the total value of the portfolio is $S(T) + \max(K - S(T), 0) = \max(S(T), K)$.

Thus the two portfolios have equal value at time T so they must have equal value at time 0, and hence

$$C + Ke^{-rT} = P + S(0)e^{-qT}.$$

Another case is when the precise present value of the dividends to be paid before time T, denoted D, is known. In this case Portfolio A should consist of one call option plus $Ke^{-rT} + D$ in cash, while portfolio B consists of one put option plus 1 share. Then since portfolio B accrues dividends with present value D and accumulated value De^{rT} by time T, both portfolios have value $\max(S(T), K) + De^{rT}$ by time T, so they have the same value at time 0, so

$$C + Ke^{-rT} + D = P + S(0)$$