10. Bonds and swaps

10.1 Zero coupon bonds A zero coupon bond is a contract available from the Government, promising to pay a specified amount K at a future time T. These are offered at time zero (the present) at a given price $P(0,T)$. According to the formula for time-value of money, we should have

$$P(0,T) = e^{-rT}K$$

In practice, K and $P(0,T)$ are known and we solve for r. The resulting r is the *zero-coupon yield*. (in general, for a given bond the *yield* is the interest rate implied by the payment(s) specified on a bond). Solving the above equation gives us a zero-coupon yield of

$$r = \frac{\log(K/P(0,T))}{T}$$

In practice, the interest rate (yield) implied by the bond is not the same for all T so to emphasize this we write $Z(0,T)$ for the above value of r, i.e. the zero-coupon yield for a bond issued at time zero with maturity (payoff) at time T. We have

$$P(0,T) = e^{-TZ(0,T)}K;$$
$$Z(0,T) = \frac{\log(K/P(0,T))}{T}$$

A plot of $Z(0,T)$ against T is called the *yield curve*. Typically, it slopes upwards because for longer-term loans the lender is rewarded with a greater yield for loss of liquidity.

10.2 Coupon-bearing bonds. The zero-coupon rates are not directly observable for all times T. This is because zero-coupon bonds are not available at all possible maturities. Typically, they are available at at most four maturity dates per year, and only up to two years or so into the future. Government bonds with longer maturities than two years usually come with *coupon payments* included before the maturity date, and these can be used to compute yield rates $Z(0,T)$ for larger values of T.

Coupon-bearing bonds (also called fixed-interest securities) are offered with a *nominal value* (which need not be the same as the price for which the bond is sold) and a *coupon rate*. Payments called *coupons* are made, based on the nominal value and the coupon rate, at intervals up to the maturity date at which the bond is redeemed, i.e. the final payment is made. The redemption price is specified in advance, and in many cases is the same as the nominal amount (*redemption at par*).

Consider for example a bond with nominal value £100, maturity date in 3 years' time, with coupon rate 6% payable semiannually and redemption at par. Suppose this bond is currently available for £98. This bond amounts to a promise of payments of £3 every six months for three years with an additional payment of £100 after 3 years. So overall the payments are £3 at times $(0.5, 1, 1.5, 2, 2.5)$ and £103 at time 3.

One quantity of interest for such a bond is the *yield*. This is the choice of r based on which the total present value of the payments is equal to the price at which the bond is being sold. In our example, the yield rate is the value of r which solves the equation

$$98 = 100e^{-3r} + 3\sum_{i=1}^{6} e^{-ri/2}.$$

In general, if the coupon payments are of amount C at times $t_1, \ldots, t_n$ and the redemption payment is A at time T (often with $T = t_n$), and the price of the bond is P, then the yield equation is

$$P = Ae^{-rT} + C\sum_{i=1}^{n} e^{-rt_i}.$$

Equations such as this cannot usually be solved analytically unless the number of payments is small, but can be solved numerically (e.g. using Newton-Raphson). If $n = 2$ and $T = t_2$ and $t_2 = 2t_1$ then the yield can be found by solving a quadratic.

Since a bond such as the above has payments over several time frames, the yield is not directly useful in calculating zero-coupon yields. However, if we already know zero-coupon yields for all the time-frames at which coupon payments are being made, then we can compute the zero-coupon yield for the maturity date of the bond based on the price of the bond, using the equation

$$P = Ae^{-Z(0,T)T} + C\sum_{i=1}^{n} e^{-Z(0,t_i)t_i}$$

For example, if in the above example we already knew the zero-coupon rate $Z(0, T)$ was 0.05 for $T = 0.5, T = 1, T = 1.5$, and 0.06 for $T = 2$ and $T = 2.5$ we could compute $Z(0, 3)$.

Indeed, using the payments for the bond described above, and the information just given, with $r = Z(0, 3)$ we'd have

$$\begin{aligned} 98 = 3(&e^{-0.05\times.5} + e^{-0.05\times1} + e^{-.05\times1.5} \\ &+e^{-0.06\times2} + e^{-0.06\times2.5}) + 103e^{-3r} \end{aligned}$$

and solve for r to get $r = Z(0, 3)$. Rearranging the above gives

$$\begin{aligned} 103e^{-3r} &= 98 - 3(e^{-0.025} + e^{-0.05} + e^{-.075} \\ &\qquad +e^{-0.12} + e^{-0.15}) = 84.19427 \end{aligned}$$

and hence $e^{3r} = 103/84.19427$ so that $r = (1/3)\log(103/84.19427)$ so that $Z(0, 3) = r = 0.0672$

If the above values of $Z(0, t)$ were the only ones directly computable from Government bonds, we could estimate $Z(0, t)$ for intermediate times by linear interpolation. For example,

$$Z(0, 2.75) = \frac{Z(0, 2.5) + Z(0, 3)}{2} = \frac{0.06 + 0.0672}{2}$$

so that $Z(0, 2.75) = 0.0636$. Thus we could sketch the yield curve from $T = 0.5$ up to $T = 3$, based on the information given.

10.3 Forward rates. Recall that we used notation $Z(0, t)$ for the zero-coupon rate, i.e. the rate compounded continuously on a loan taken out now (at time 0) that is repaid as a lump sum at time t. In other words, a loan of 1 at time 0 with agreed repayment date t is repaid by amount $e^{tZ(0,t)}$ at that time.

The *forward rate* $f(0,t,T)$ (where $0 < t < T$) is the corresponding rate for a loan which is agreed upon now (at time 0) but where the actual loan takes place at time t and is repaid in one go at time T. That is, if it is agreed that a loan of 1 is to be made at time t and repaid at time T, the agreed size of the repayment is

$$e^{(T-t)f(0,t,T)}.$$

Arbitrage arguments suggest that we should have

$$e^{tZ(0,t)}e^{(T-t)f(0,t,T)} = e^{TZ(0,T)}. \tag{1}$$

For example, if we had $e^{tZ(0,t)}e^{f(0,t,T)(T-t)} < e^{TZ(0,T)}$, then you could borrow amount 1 now, agreeing to repay $e^{tZ(0,t)}$ at time t, and agree (now) to borrow amount $e^{tZ(0,t)}$ at time t with repayment $e^{tZ(0,t)}e^{(T-t)f(0,t,T)}$ at time T. At the same time you could lend 1 now with an agreement to receive $e^{TZ(0,T)}$ at time T. Your net initial outlay is zero, and you fund the repayment of the first loan at time t using the second loan, and at time T you end up with a profit, i.e. arbitrage.

Using equation (1), we obtain

$$f(0,t,T) = \frac{TZ(0,T) - tZ(0,t)}{T-t}.$$

In other words, the forward rates can be obtained from the zero-coupon rates. For example, if $Z(0,1)$ was 6% and $Z(0,2)$ was 7% (both compounded continuously as usual) then

$$f(0,1,2) = \frac{2 \times 0.07 - (1 \times 0.06)}{2-1} = 0.08$$

which is reasonable since then the rate for the two years (7%) is the average of the rate for the first year (6%) and the rate for the second year (8%).

10.4 Floating interest rates. If you acquire a short-term loan at a given time or make a short-term risk-free investment, the interest rate on the loan is called the 'floating interest rate'. A good measure of the floating interest rate is the London Inter-bank Offer Rate (LIBOR) which is available on 1-month deposits, 3-month deposits and so on. If we denote the LIBOR for a period of τ starting at time t as $R(t,\tau)$ compounded continuously then a unit investment at time t accumulates to $e^{R(t,\tau)\tau}$ by time $t+\tau$. In practice, the LIBOR for a period of time τ is often quoted as being compounded with the same time-span τ rather than continuously.

If a company acquires a loan from a financial institution at a floating rate, the rate is usually quoted as LIBOR plus a percentage. The percentage added depends on the credit rating of the company. For example if the interest on a loan of 100 is paid every 3 months, and the three-month LIBOR is 12 per cent compounded quarterly, and the loan is at LIBOR plus 3 per cent compounded quarterly, then the interest payable after 3 months would be $(100) \times ((.12 + .03)/4) = 15/4$. Just after that payment the amount due would still be 100 and the next payment, assuming non-repayment of the loan, would be calculated based on the new LIBOR.

10.5 Interest rate swaps. A *swap* is an exchange between two parties of two cash flows where the cash flows are determined by two different underlying variables.

An interest rate swap is an exchange of loans between two parties A and B (often one of these parties is a financial institution) whereby the principal and time-frames of the two loans are the same, but one loan is paid back at a fixed rate and the other is paid back at a floating rate.

For example, a swap could involve A and B lending each other 100 (million pounds) over a 5-year period, with semiannual repayments and A pays B at a fixed interest rate of ρ (Greek letter rho) compounded continuously, while B repays A at a the 6-month LIBOR rate.

The payments (in both directions) are just to 'service' the loans and not to repay the principal. The principal is returned at the end of the 5-year period but since the principal payments cancel out, there is no cash payment at the end, except the final interest payment.

If the contract is entered into at time 0, then the payment at times $t = 0.5, 1, 1.5, \ldots, 4.5, 5$.

At each time $t = 0, 0.5, \ldots, 4.5$ the LIBOR at that time is ascertained, which we are calling $R(t, 0.5)$ compounded continuously, and the net payment from A to B at time $t + 0.5$ is

$$100((e^{\rho/2} - 1) - (e^{0.5R(t,0.5)} - 1))$$

The value of ρ (sometimes called the 'swap rate') is computed so that no upfront payment is needed. How do we do this? We consider the general case where the principal is of amount K and interest payments are at times $t_1, \ldots, t_n$ (assumed equally spaced). The 'virtual repayments' of principal are at time T, normally with $T = t_n$.

Consider the two cash flows separately. First, the payments from A to B. Then A pays B an amount k at each time t_i plus a 'virtual principal repayment' of K at expiry. We would like to know k (in the above example k would be $100(e^{\rho/2} - 1)$ and we'd like to know ρ).

Assuming the contract starts at time zero, the total present value of the payments from A to B is

$$k \sum_{i=1}^{n} e^{-Z(0,t_i)t_i} + Ke^{-Z(0,T)T}$$

where $Z(0, t)$ is the zero-coupon rate discussed earlier.

The total present value of the payments from B to A is simply K. Indeed, if I had amount K right now and invested it at the floating rate until time t_1, I would receive precisely the first payment from B to A at time t_1. If I then kept the payment but reinvested the K from time t_1 to t_2, I'd receive at time t_2 an amount equal to the second payment from B to A. Continuing in this way, and keeping my investment K as well as the interest payment at the end, we can see that the initial investment K generates a cash-flow which exactly matches the payments from B to A, and arbitrage arguments tell us K must be the present value of these payments. Hence for the payments to match we need

$$k \sum_{i=1}^{n} e^{-Z(0,t_i)t_i} + Ke^{-Z(0,T)T} = K$$

so that

$$k = K \frac{1 - e^{-Z(0,T)T}}{\sum_{i=1}^{n} e^{-Z(0,t_i)t_i}}.$$

In the earlier example, we'd have

$$k = 100(e^{\rho/2} - 1) = 100 \frac{1 - e^{-5Z(0,5)}}{\sum_{i=1}^{10} e^{-(i/2)Z(0,i/2)}}$$

and from this we could compute ρ if we wished.

10.6 Valuation of swaps during their term. As just described, the initial value of a swap contract is zero, but at intermediate times it might not be zero. Computing the value of a swap at an intermediate time may be of interest to a financial institution assessing its credit risk, i.e. exposure to the possibility of a company defaulting on a swap.

Suppose we are at time 0 now and the swap contract started at some earlier time but otherwise is as above. Assume the expiry date T is positive and let $t_1, t_2, \ldots, t_n$ denote the remaining payment dates, i.e. those dates for repayment which fall after time 0, arranged in increasing order.

The present value of the remaining payments from A to B is

$$k \sum_{i=1}^{n} e^{-Z(0,t_i)t_i} + K e^{-Z(0,T)T}$$

just as before. The present value of the remaining payments from B to A just after time t_1 (i.e. just after the next payment) will be precisely K by the same argument as before. Adding the payment (of amount k^*, say, which is already known) at time t_1 to this we see the present value of the remaining payments from B to A just BEFORE time t_1 will be $K + k^*$, and discounting to time zero we find the present value of remaining payments from B to A is

$$(k^* + K)e^{-Z(0,t_1)t_1},$$

and the total present value to B of the swap contract is

$$k \sum_{i=1}^{n} e^{-Z(0,t_i)t_i} + K e^{-Z(0,T)T} - (k^* + K)e^{-Z(0,t_1)t_1},$$

which could be positive or negative according to circumstance.

For example, consider the 5-year swap described earlier. What is its value after 2.75 years? Then we have 2.25 years left, so taking time zero to be the present (i.e. 2.75 years after the start of the swap) the value to B of the swap is

$$k \sum_{i=0}^{4} e^{-Z(0,0.25+(i/2))(0.25+(i/2))}$$
$$+100e^{-2.25Z(0,2.25)} - (k^* + 100)e^{-Z(0,0.25)0.25},$$

where k^* was determined three months ago as 100 times half the six-month LIBOR compounded semiannually.

10.7 Motivation for interest rate swaps. The main motivation for the interest rate swap market is differences in the credit quality between entities borrowing money. Specifically, some companies may have a better access to short-term funding than to long-term funding; namely, companies with a poor credit rating. Interest rate swaps allow companies to take advantage of the global markets more efficiently by bringing together two parties that have an advantage in different markets.

Consider an example. Company Z has relatively poor credit rating A and looks for a loan to finance a new long-term project. Assume that two types of loan are available:

1. a sequence of short-term loans with variable interest rates, which generates a sequence of uncertain different future payments $C_1, \ldots, C_n$, or

2. long-term loans with fixed interest rate, future payments are equal to each other and known, but the fixed interest rate is high.

Assume that company Z can borrow at 11.2% in the fixed-rate market and at at LIBOR+1% in the floating-rate market. *Problem for the company: to find a cheaper loan with a fixed interest rate.*

Consider another company X which is highly rated, with credit rating AAA. X has a higher rating, hence can borrow at better rates in both fixed-rate and floating-rate markets. Assume X can borrow at 10% in the fixed-rate market and at LIBOR+0.3% in the floating-rate market.

If X seeks floating rate funds, then X and Z can *swap*. Both X and Z borrow the same amount as follows:

- X borrows at 10% in the fixed rate market

- Z borrows at LIBOR+1% in the floating rate market,

then they *swap*:

- X pays a fixed interest rate 10% back (outside) and floating LIBOR to Z and receives fixed interest rate R% from Z

- Z pays a floating LIBOR+1% back (outside) and a fixed interest rate R% to X and receives LIBOR from X

Overall: X pays a floating interest rate $LIBOR + (10 - R)\%$ and Z pays a fixed interest rate $(R + 1)\%$. To arrange for both parties to get the same percentage advantage from the deal, R would determined from the following equation

$$LIBOR + 0.3 - (LIBOR + 10 - R) = 11.2 - (R + 1).$$

Solving this for R we would obtain $R = 9.95\%$. As a result of the swap:

- Z get a loan at the fixed interest rate 10.95%

- X get a loan at the floating interest rate LIBOR+0.05%.

Moreover, company Z hedges: it pays fixed known beforehand amounts instead of paying floating.

The preceding argument is based on *comparative advantage argument*. The borrowing spread between Z and X is greater for fixed-rate funding (1.2%) than for floating-rate funding (0.7%). It is said in this situation that X *has a comparative advantage at the fixed rate market and* Z *has a comparative advantage at the floating rate market.* It does not mean that Z pays less than X at this market; it means that the extra amount that Z pays over the amount paid by X is less in this market. In other words, Z has a *lower comparative disadvantage* relative to X in the floating rate market.

The choice of R above gives an equal benefit to both X and Z, though it is not necessarily the same as what would be determined by the arguments of Section 10.4 which were based on arbitrage.

10.8 Currency Swaps. In a currency swap, two parties agree to exchange bonds in two different currencies, where the bonds are of approximately equal value and equal duration, and for each bond the interest payments are made in the currency of that bond at a pre-specified interest rate (in general, not the same rate for both bonds). For example, A and B might engage in a currency swap whereby initially A pays B a certain amount K in dollars while B pays A a certain amount K' in pounds. Then at times $t_1, \ldots, t_n$ B pays A interest payments (typically semiannually) in dollars on the notional principal of K, using an interest rate of R, while A pays B interest payments in pounds at a rate R'. At the conclusion of the swap (time T), B repays A the notional principal of K dollars while A repays B the notional principal of K' pounds. The interest rates R and R' could be both fixed in advance, or could be floating, or could be one of each depending on the nature of the swap.

Consider, for example, a yen/dollar currency swap between companies where company A pays 5% per annum (with annual compounding) on a 1200 million yen loan while company B pays 8% with annual compounding on a \$10 million loan. Suppose the swap has 3 years to run, and the current exchange rate is 110 yen per dollar, and interest rates with continuous compounding are assumed to be constant at 9% per annum for dollars and 4% for yen. What is the value of the swap to company B?

Answer: The total present value in yen of remaining payments from A to B is

$$(60e^{-0.04} + 60e^{-0.08} + 1260e^{-0.12}) \text{ million yen}$$

which comes to 1230.55 million yen, and converting to dollars this comes to 1230.55/110 million dollars, or 11.19 million dollars.

The total present value in dollars of remaining payments from B to A is

$$\$(0.8e^{-0.09} + 0.8e^{-0.18} + 10.8e^{-0.27}) \text{ million}$$

which comes to \$9.64 million. So the total value of the contract to B is $11.19 - 9.64 = 1.55$ million dollars.