MA50196 Financial Derivatives

Mathew Penrose
Department of Mathematical Sciences
University of Bath
Bath BA2 7AY
masmdp@bath.ac.uk
http://www.maths.bath.ac.uk/∼masmdp/
Office: 4W4.11

Semester 1 2019-20

1. Introduction

1.1 overview

The main subject of this course is the evaluation of promised future payment(s), for example:

1. A Government bond (zero-coupon bond) paying a fixed amount at a future time T.
2. A bond promising to make a series of payments (coupons) at times $t_1, t_2, \ldots, t_n$
3. A zero-coupon bond issued by a small company.
4. A futures contract in a foreign currency, involving a payment at time T in that currency.
5. A futures contract in a stock in a certain company, involving a transfer of that stock at time T.
6. A stock option, where the payment at time T depends on the value of the stock at that time.

The two main sources of difficulty in valuation of these future payments are:
(a) Interest rates and the time value of money.
(b) Uncertainty over the level of payment.

Factor (a) is simplified if we assume a constant interest rate, though this might not be a valid assumption. In factor (b), the uncertainty is often tied to a particular source of uncertainty (e.g. future exchange rates, or the price of a certain stock).

A secondary theme of the course is how one can protect oneself against these uncertainties.

Because of these uncertainties, the mathematical theory of probability is needed, along with a certain amount of calculus.

As the course progresses, we shall go into more detail on the valuation of futures contracts, and on the various types of options available.

After introducing the necessary mathematical theory, we shall proceed via the binomial pricing model to the celebrated Black-Scholes formula for option pricing.

We then discuss elements of risk management (the "Greek Letters"). We shall conclude with some discussion of bonds and swaps.

Recommended books on Mathematical Finance

- J.C.Hull: Options, Futures and Other Derivatives.
- J.C.Hull: Fundamentals of Futures and Options Markets.

Other related books

- P. Wilmott: Paul Wilmott Introduces Quantitative Finance.
- Desmond J. Higham: An introduction to financial option valuation. (Quite cheap. Possibly does not match the course as well as the others).

1.2 Hedging, forwards and futures contracts

Often, a business faces uncertainty over its future revenues and/or costs. For example:
A farmer does not know the price he will get for next year's wheat crop.
An airline is uncertain over future costs of fuel.

Typically, companies (and individuals) would like to **hedge**, i.e. to reduce or eliminate uncertainty. (Individuals do this when they buy insurance).

The farmer could hedge by entering into a **forward** or **futures** contract whereby he agrees now to sell his crop in the future for some price which is fixed now. (In this case we say the farmer takes the *short position* in the commodity in question, in this case wheat).

The airline, similarly, could hedge by taking a *long* position in a futures contract in airline fuel (or, if such a contract is unavailable, in some proxy such as oil). In other words, the airline could agree now to buy the commodity at a fixed future date, for a price which is fixed now.

Both Forward and Futures contracts involve commitment between parties to a future exchange of a certain asset (e.g. wheat, oil, stocks) at a future time T for a certain price K (the *strike price*).

What is the difference between forward and futures contracts? A forward contract is generally an over the counter (OTC) agreement between two individuals or businesses. A futures contract is more formal, and is regulated by an exchange (e.g. the Chicago Mercantile Exchange) which gurantees the payments. Futures contracts are generally available with certain specified payment dates (e.g. 4 specified dates per year might be available); moreover, in futures contracts the value of K is chosen so that no upfront payment is required by either party, whereas this might not be the case for forwards contracts (more on this later).

In a forward or futures contract, the **payoff** at time T for the holder of the long position is $S(T) - K$, where $S(T)$ is the value of the asset at time T. For the holder of the short position, conversely, the payoff at time T is $K - S(T)$.

1.3 Options

Modern financial exchanges offer not only futures but also a great many other **derivatives** with payoffs whose value is determined by (i.e. derived from) the price at a future time of a certain commodity (or stock, or stock index, or exchange rate).

A classic example of a derivative is an **option**, which is a contract between two parties (A and B say) giving the option holder (A) the right but not the obligation, to buy the asset from the other party (B) at time T (the maturity date) for price K (the strike price).

The above option is called a **call** option. Conversely, a **put** option gives A the right but not the obligation to sell the asset to B for price K at time T.

These options are called **European** options because the exchange of asset for cash (if A chooses to exercise the option) takes place at a fixed time T. There are other options, where this exchange could takes place at some other time. For example, in an *American* option, the option holder (A) has the right to exercise the option at any time t with $0 \leq t \leq T$.

What is the payoff for a European Call option? The option holder (A) will not exercise the option unless $S(T) > K$, in which case (s)he gets a payoff of $S(T) - K$. If $S(T) < K$, A does not exercise the option and gets a payoff of zero. Combining these, the payoff to A is

$$\max(S(T) - K, 0)$$

For any number x, we use the notation x^+ to denote the number $\max(x, 0)$. With this notation, the payoff to A at time T for the call option can be written as

$$(S(T) - K)^+$$

Conversely, with a European put option, the payoff to A is

$$(K - S(T))^+$$

In both cases the payoff cannot be negative. Clearly the option is of some value to A in both cases, so A can expect to have to pay B a certain amount upfront now for the option. Finding a 'suitable' upfront amount to pay is a major part of this course.

1.4 Absence of arbitrage and other assumptions

If an individual or company can carry out a series of transactions whereby they ultimately achieve a risk-free profit, they are said to achieve **arbitrage**.

For example, suppose a stock is traded both in New York and London. Suppose the stock price is 175 dollars in New York and 100 pounds in London, and that the exchange rate is 1.8 dollars per pound. Then an arbitrageur could buy the stock in New York, sell it in London, and make a profit of (180-175)=5 dollars per share.

It is assumed that if such an arbitrage opportunity existed, arbitrageurs would make the above deals and drive up the price in New York, driving down the price in London, and so the arbitrage opportunity would soon be eliminated. Thus it is generally assumed that **in an efficient market there are no arbitrage opportunities**.

Some other assumptions which are often made (some of which are needed for the above arbitrage opportunity to be genuine).

1. No transactions costs in buying or selling shares, or currency.
2. Individual traders are *market takers* (rather than market makers), i.e. their actions do not affect prices.
3. Stock and other assets can be bought sold in arbitrary amounts, including fractional amounts.
4. Traders can short sell, i.e. sell stock or other assets without acquiring it first.

1.5 Time value of money

If you invest amount A_0 into a bank account at time $t = 0$ at interest rate R per annum, what is the value $A(t)$ of your investment at time $t > 0$?

The answer depends on how the interest is compounded, i.e. how often the interest is added to the investment. It is assumed here that all interest payments are reinvested in the account i.e. that we are using *compound interest*.

If interest is compounded *annually*, then after one year the interest added is A_0R, so that

$$A(1) = A_0 + A_0R = A_0(1 + R)$$

and since this amount is reinvested at time 1, and earns interest at rate R once more for the duration of the second year, we have

$$\begin{aligned} A(2) = A(1)(1 + R) &= A_0(1 + R)(1 + R) \\ &= A_0(1 + R)^2 \end{aligned}$$

and repeating the argument, we have for any positive integer (whole number) t that

$$A(t) = A_0(1 + R)^t.$$

Now suppose interest is compounded *semi-annually*, i.e. every six months. Then each half-year, interest is added to the account with a constant of proportionality $R/2$ (we divide by two because it is only for half a year). For example, an investment of £100 for six months at a rate of 8% compounded semiannually, accrues interest of £4 after six months.

In general, if interest is compounded semiannually, we have

$$A(1/2) = A_0 + A_0(R/2) = A_0(1 + R/2).$$

Because of the compounding, at time 1 we obtain

$$A(1) = A(1/2)(1 + R/2) = A_0(1 + R/2)^2.$$

Multiplying this out note that we obtain $A(1) = A_0(1 + R + R^2/4)$, and that if $R > 0$ this is more than in the case where interest is compounded annually. This is because with semiannual compounding, the interest from the first six months itself earns interest during the second six months.

Repeating the above argument n times, we have for integer n that

$$A(n/2) = A_0(1 + R/2)^n$$

or in other words, for t a multiple of $1/2$ we have

$$A(t) = A_0(1 + R/2)^{2t}.$$

Now suppose interest is compounded m times per year. Then

$$A(1/m) = A_0(1 + R/m)$$

and for integer n we have

$$A(n/m) = A_0(1 + R/m)^n.$$

In other words, for t a multiple of $1/m$ we have

$$A(t) = A_0(1 + R/m)^{mt} = A_0((1 + R/m)^m)^t.$$

In fact, for our purposes $A(t)$ will normally be calculated on the assumption that interest is compounded *continuously*, i.e. based on the assumption that compounding takes place all the time. This just amounts to taking the limit as m becomes large in the above formula. Using the 'well-known' mathematical identity

$$\lim_{m\to\infty}(1 + \frac{x}{m})^m = e^x$$

(where $e \approx 2.71828$ is a well-known mathematical constant) or less formally

$$(1 + \frac{x}{m})^m \approx e^x, \quad \text{for } m \text{ large}$$

We find by that in the limit of very large m we have $(1 + R/m)^m = e^R$ so that

$$A(t) = A_0((1 + R/m)^m)^t = A_0(e^R)^t = A_0e^{Rt}$$

One advantage of compounding continuously is that we do not have to worry about computing $A(t)$ for time-periods t which are not a multiple of the period of compounding.

In fact, we'll usually write r not R for the rate of interest compounded continuously, so that

$$A(t) = A_0e^{rt}$$

Another way to see this formula is via differential equations. If the rate of interest, continuously compounded, is r then for small h we have

$$A(t + h) \approx A(t)(1 + rh)$$

so that

$$\frac{d}{dt}A(t) = rA(t)$$

and the solution of this differential equation with $A(0) = A_0$ is

$$A(t) = A_0 e^{rt}$$

as before. The factor of e^{rt} is called an *accumulation factor*.

What is the value at time 0 of a promised payment of amount K at time T in the future (e.g. a zero-coupon bond, or a payment as part of a futures contract)? Assume no risk of default.

If you are promised such a payment, you could borrow amount B at time zero, and use the payment you receive at time T to pay off your debt. This will work for some suitable choice of B. To see which choice, note that you will have to repay Be^{rT} at time T, so B needs to satisfy

$$Be^{rT} = K, \quad \text{i.e.} \quad B = Ke^{-rT}.$$

Thus based on the future payment K, the amount you can acquire now is this value of B, which we call *present value* of the promised payment at time T. That is, the present value of the payment of K at time T is Ke^{-rT}. We call the factor of e^{-rT} the *discount factor*.

In the above we have assumed that the interest rate r is a known constant. In practice, r can vary with time, though we will often assume it is constant. Also, in practice r needs to be deduced from the values of certain bonds available on the market, as we discuss later.