**MA50196: Sheet 9**

1. Suppose the risk-free rate of interest is $r$ compounded continuously. Suppose $f(t,s)$ is a function of two variables that is twice continuously differentiable. Suppose a stock price process $S_t$ follows a log-normal process with infinitesimal drift $\mu$ and volatility $\sigma$.

   (a) Starting from the formula $dS = \mu S dt + \sigma S dZ$, briefly derive Ito's formula for the increment $df(t, S_t)$ over a small time-increment $dt$, in terms of $dt$ and the change in stock price $dS_t$ (you may use the rule $(dZ)^2 = dt$ without proof).

   (b) Use the formula from the preceding part to show that if $f(t,s)$ denotes the value of a derivative at time $t$ if $S_t = s$, for a European-style derivative with payoff $F(S_T)$ at time $T$, then $f(t,s)$ satisfies the Black-Scholes differential equation
   $$\frac{\partial f(t,s)}{\partial t} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 f(t,s)}{\partial s^2} + rs\frac{\partial f(t,s)}{\partial s} = rf(t,s)$$
   on $t \leq T, s > 0$, with boundary condition $f(T,s) = F(s)$ for all $s > 0$.

   (c) Briefly explain why, for $x > 0$, the solution to this differential equation satisfies $f(0,x) = e^{-rT}E[F(\tilde{S}_T)]$ where $\tilde{S}_t$ is a log-normal process with infinitesimal drift $r$ and volatility $\sigma$, starting from $\tilde{S}_0 = x$.

2. Assume that a trader has a portfolio of financial derivatives which all depend on a price of the same underlying asset.

   a) Suppose the price of the asset immediately increased by 0.34 and all other variables remained unchanged. Estimate profit/loss which the trader can realize from this price change if the portfolio delta is currently equal to 21000.

   b) How does the result of question a) change if the portfolio gamma is also known and equal to $-5000$?

3. Assume that a portfolio of options on gold has $\Delta > 0$. Suppose that "it is very likely" that the gold price is about to increase slightly. Also, we may assume that all other parameters influencing the option price are going to be unchanged.

   a) Would you advise someone to delta hedge the portfolio?

   b) Would you change your opinion if the portfolio delta was negative?

   c) Comment on the assumption about the gold price in the question.

4. Consider a portfolio formed by 5000 long call options on the same underlying asset. Each option has a delta of 0.47 and a gamma of 0.9.

   a) What position in another type of traded call options with $\Delta = 0.5$ and $\Gamma = 2$ (on the same underlying) does one need to add to make the portfolio gamma-neutral?

   b)What position in the underlying is needed to make the portfolio delta-neutral as well?

5. Consider a binary call option on a stock with current price $S$ and with maturity at time $T$ (current time is zero) and strike price $K$. Assume the stock price follows a log-normal process with expected rate of return $\mu$ and volatility $\sigma$. Assume the risk-free interest rate is $r$ compounded continuously.

   (a) Show that the price of the binary call option is $e^{-rT}\Phi(x_0)$, where $x_0$ is as in the Black-Scholes formula.

   (b) Show that for this binary option,
   $\Delta = \varphi(x_1)/(K\sigma\sqrt{T})$, where $x_1$ is as in the Black-Scholes formula.

   (c) Find a formula for $\Gamma$ for this option.

6. Suppose an investor has bought 1000 of 9-months call options on a certain asset, such that each option has a delta of 0.6.

   (a) What position in the underlying asset is required for delta hedging?

   (b) What position in 1-year futures contracts on the same underlying is required for delta hedging if the 1-year risk-free interest rate is 8%?