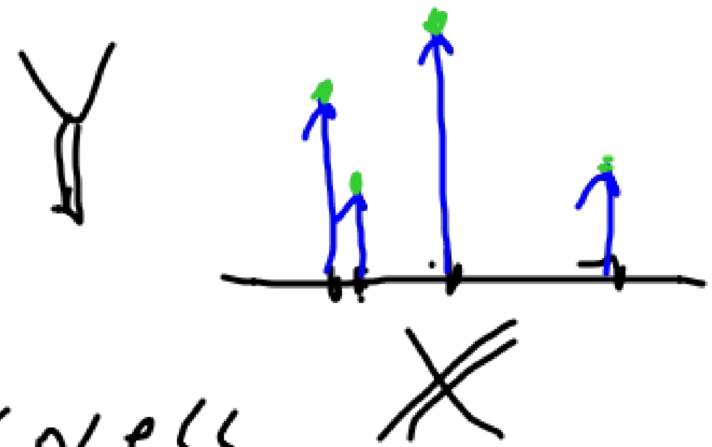
V.2 The marking Theorem

$(\mathbb{X}, \mathcal{X}), (\mathbb{Y}, \mathcal{Y})$ - measurable spaces.



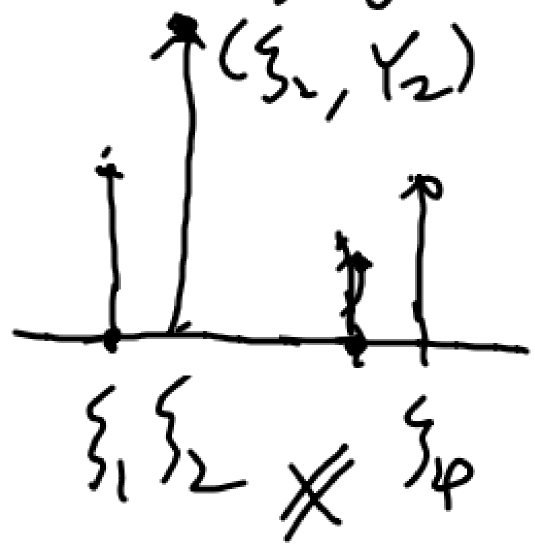
Given a proper point process

$$\eta = \sum_{i=1}^{\kappa} \delta_{\xi_i} \quad \text{in } \mathbb{X},$$

suppose each ξ_i has a $\mathbb{Y}$-valued mark Y_i attached with

$\mathcal{L}(Y_i \mid \eta)$ depending only on ξ_i:

$\mathbb{P}[Y_i \in dy \mid \eta] = K(\xi_i, dy)$ $\quad \mathbb{Y}$



$\theta := \sum_{i=1}^{\kappa} \delta_{(\xi_i, Y_i)}$ is a marked point pr. We assume

$K : \mathbb{X} \times \mathcal{Y} \to [0,1]$ is a stochastic kernel:

i.e. (i) $K(x, \cdot)$ is a prob measure on $(\mathbb{Y}, \mathcal{Y})$

(ii) $K(\cdot, A)$ is $\mathcal{X}$-measurable $\forall A \in \mathcal{Y}$.

Write $\eta = \sum_{i=1}^{K} \delta_{\xi_i}$ with $K \in \mathbb{N}_0$, $\xi_i \in \mathbb{X}$

For $k \in \mathbb{N}_0$, let

$\mathbb{P}_k$ be a choice of the prob. distribution of $(\xi_1, \xi_2, \dots)$ given $K = k$ (distribution on $\mathbb{X}^\infty$). Define

$\widetilde{\mathbb{P}}_k$ on $(\mathbb{X} \times \mathbb{Y})^\infty$ (another meas.) by

$$\widetilde{\mathbb{P}}_k(f) = \int_{\mathbb{X}^\infty} f((x_i, y_i)_1^\infty) \prod_{i=1}^{\infty} K(x_i, dy_i)\, \mathbb{P}_k(d\underline{x}).$$

Put $\theta = \sum_{i=1}^{\infty} \delta_{(\xi_i, Y_i)}$ with

$$P[(\xi_i, Y_i)_1^\infty \in \cdot\,] = \sum_{k=0}^{\infty} P[K = k]\, \widetilde{\mathbb{P}}_k(\cdot)$$

Proposition (5.4) Given θ as above, and $u \in \mathbb{R}_+(\mathbb{X}\times\mathbb{Y})$

$$L_\theta(u) = \mathbb{E}\, e^{-\theta(u)} = L_\eta(u^*), \text{ with}$$

$$u^*(x) = -\log \int_{\mathbb{Y}} e^{-u(x,y)} K(x,dy)$$

Proof

$$L_\theta(u) = \sum_{k\in[0,\infty)} \mathbb{P}[K=k] \int_{\mathbb{X}^\infty}\int_{\mathbb{Y}^k} \underbrace{e^{-\sum_{i=1}^{k} u(x_i,y_i)}}_{\prod_{i=1}^{k} \exp(-u(x_i,y_i))} \prod_{i=1}^{k} K(x_i,dy_i)\, \mathbb{P}_k(d\underline{x})$$

$$= \sum_k \mathbb{P}[K=k] \int \left(\prod_{i=1}^{k} e^{-u^*(x_i)}\right) \mathbb{P}_k(d\underline{x})$$

$$= \mathbb{E}\left[e^{-\eta(u^*)}\right] = L_\eta(u^*). \quad \blacksquare$$

Theorem (5.5) Suppose η is a PPP in $\mathbb{X}$ with σ-finite mean measure λ. Then Θ is a PPP in $\mathbb{X}\times\mathbb{Y}$ with mean measure $\lambda\otimes K$:

$$\lambda\otimes K(f) = \int_{\mathbb{X}}\int_{\mathbb{Y}} f(x,y)\,K(x,dy)\,\lambda(dx)$$

$[f\in\mathcal{R}_+(\mathbb{X}\times\mathbb{Y})]$

<u>Proof</u> Suppose $\lambda(\mathbb{X}) < \infty$. Then WLOG

$$\eta = \sum_{i=1}^{K} \delta_{X_i} \quad \text{with}$$

$$K \sim Po(\gamma), \; X_i \sim Q = \frac{\lambda(\cdot)}{\gamma}, \; \gamma = \lambda(\mathbb{X})$$

(indep)

Then $\theta = \sum_{i=1}^{K} \delta_{(X_i, Y_i)}$ with $(X_i, Y_i) \sim Q \otimes K$ indep.

By proof of prop. III.2, θ is a PPP with intensity

$$\gamma \underbrace{Q \otimes K}_{\text{prob. meas.}} = \lambda \otimes K.$$

If $\lambda(\mathbb{X}) = \infty$, write

$\lambda = \sum_{i=1}^{\infty} \lambda_i$, $\lambda_i(\mathbb{X}) < \infty$ and

$\eta = \sum_{i=1}^{\infty} \eta_i$ $\quad \eta_i$ PPP (λ_i) (indep)

Let θ_i be the K-marking of η_i

$\theta = \sum_{i=1}^{\infty} \theta_i$ is a PPP with intensity

$\sum_{i=1}^{\infty} \lambda_i \otimes K = \lambda \otimes K$. $\square$

V.3 Thinning/colouring

__Proposition__ (Thm 5.2: restriction Thm)

Let η be a PPP in a measurable space $(\mathbb{X}, \mathcal{X})$ with s-finite intensity measure λ. Let $C_1, C_2, \dots \in \mathcal{X}$ be pairwise disjoint. Then $\eta_{C_1}, \eta_{C_2}, \dots$ are indep. PPPs with mean measure $\lambda_{C_1}, \lambda_{C_2}, \dots$ respectively, where $\mu_C(A) = \mu(C \cap A)$, $\forall A, C \in \mathcal{X}$, μ a measure on $(\mathbb{X}, \mathcal{X})$

Proof If $\eta \overset{d}{=} \eta'$ & result holds for η', it holds for η.

WLOG $\bigcup_{i=1}^{\infty} C_i = \mathbb{X}$. Let $\eta_1, \eta_2, \ldots$ be indep. PPPs with mean measure $\lambda_{C_1}, \lambda_{C_2}, \ldots$ resp. Set $\eta' = \sum_{i=1}^{\infty} \eta_i$

By superposition η' is a PPP with intensity $\sum_{i=1}^{\infty} \lambda_{C_i} = \lambda$, so $\eta' \overset{d}{=} \eta$. Also

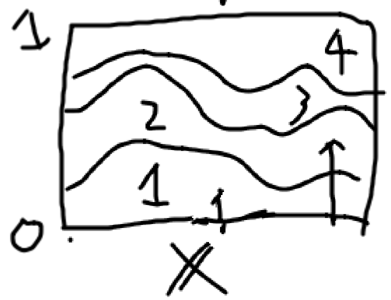
$\eta'_{C_i} = \eta_i$ So $\eta'_{C_1}, \eta'_{C_2} \ldots$ are indep PPPs with mean measures $\lambda_{C_1}, \lambda_{C_2} \ldots$ resp. $\blacksquare$

Theorem (5.8 - colouring Thm)

Suppose for $i \in \mathbb{N}$, $p_i : \mathbb{X} \to [0,1]$ is measurable with $\sum_{i=1}^{\infty} p_i(x) = 1$ and η is a PPP on $\mathbb{X}$ with s-finite mean measure λ. Let K be the stochastic kernel from $\mathbb{X}$ to $\mathbb{N}$:

$K(x, \{j\}) = p_j(x)$, $j \in \mathbb{N}$. Let θ be the K-marking of η, and

$\eta_i = \theta(\cdot \times \{i\})$. Then $\eta_1, \eta_2, \ldots$ are indep PPPs on $\mathbb{X}$, η_i having mean measure $p_i(x)\lambda(dx)$



Proof By Thm V.2, θ is a PPP in $\mathbb{X}\times\mathbb{N}$ with mean measure $\lambda\otimes K$. By Prop. V.3,

$$\theta_{\mathbb{X}\times\{1\}},\ \theta_{\mathbb{X}\times\{2\}},\ \theta_{\mathbb{X}\times\{3\}},\ \dots$$

are indep PPPs. $\theta_{\mathbb{X}\times\{i\}}$ has mean meas. $(\lambda\otimes K)_{\mathbb{X}\otimes\{i\}}$. By defn., $\forall A\in\mathcal{X}$, $i\in\mathbb{N}$

$$\theta_{\mathbb{X}\times\{i\}}(A\times\{i\})=\eta_i(A)$$

$$(\lambda\otimes K)_{\mathbb{X}\times\{i\}}(A\times\{i\})=\int_A \rho_i(x)\ \lambda(dx)$$

So $\eta_1,\eta_2,\dots$ are indep PPPs with mean measures $\rho_i(x)\,\lambda(dx)$. $\boxtimes$

$\underline{\text{VI}}$ Stationary pt processes (8)

$\underline{\text{VI}}$.1 Stationarity (8.1)

Let $d\in\mathbb{N}$. Consider pt processes in $\mathbb{X}=\mathbb{R}^d$ ($\mathcal{X}$ = Borel σ-alg. $\mathcal{B}^d$). Given $y\in\mathbb{R}^d$, define $y\in\mathbb{R}^d$, define $\theta_y:\underline{N}\to\underline{N}$

$$\theta_y\,\mu(B)=\mu(B+y),\quad B\in\mathcal{X},$$

where

$$B+y=\{b+y : b\in B\}\quad [\text{also } B-y=B+(-y)]$$

e.g. $\theta_y\,\delta_y=\delta_0$ $\quad (0=(0,\dots,0))$

$$\theta_y\,\delta_y(B)=\delta_y(B+y)=\delta_0(B)$$

Note the flow property

$$\theta_y \circ \theta_x = \theta_{x+y}, \quad x, y \in \mathbb{R}^d.$$

Defn. A pt. pr. η in $\mathbb{R}^d$ is said to be stationary if

$$\theta_x \eta \overset{d}{=} \eta \quad \forall x \in \mathbb{R}^d.$$

Prop. A (8.2) Suppose η is a stationary pt. pr. in $\mathbb{R}^d$ with $\gamma := \mathbb{E}\,\eta([0,1]^d) < \infty$. Then the intensity of η is $\gamma \lambda_d$, where $\lambda_d :=$ Lebesgue meas. on $\mathbb{R}^d$.

Proof $\forall B \in \mathcal{X}$ set

$\lambda(B) = \mathbb{E}\,\eta(B)$. For $x \in \mathbb{X}$

$$\lambda(B) = \mathbb{E}\,\theta_x \eta(B)$$
$$= \mathbb{E}\,\eta(B+x) = \lambda(B+x)$$

So λ is a translation invariant (TI) measure on $\mathbb{R}^d$ with $\lambda([0,1]^d) = \gamma$.

So $\lambda = \gamma \cdot \lambda_d$. $\blacksquare$

Next a converse for PPPs.

Prop. B (8.3) Suppose

η is a PPP in $\mathbb{R}^d$ with mean measure $\gamma \cdot \lambda_d$, some $\gamma \in [0,\infty)$.

Then η is stationary.

Proof. Let $u \in \mathbb{R}^d$. By Thm V.1 (mapping thm) $\theta_u \eta$ is a PPP win in $\mathbb{R}^d$ with mean measure $\theta_u(\gamma\lambda_d)$.

For $B \in \mathcal{X}$, $\theta_u(\gamma\lambda_d)(B) = \gamma\lambda_d(B+u)$

$$= \gamma\lambda_d(B).$$

So $\theta_u \eta$ is a PPP with intensity measure $\gamma\lambda_d$. So $\theta_u \eta \overset{d}{=} \eta$ by Prop. in sec II.3 ((i)$\Leftrightarrow$(ii)). $\square$