IV.2 Multivariate Mecke eqn.

Defn. Given $(\mathbb{X}, \mathcal{X})$ and $\mu \in \underline{N}(\mathbb{X})$ and $k \in \mathbb{N}$, define the k-th factorial measure $\mu^{(k)} \in \underline{N}(\mathbb{X}^k)$ as follows:

if $\mu = \sum_i \delta_{x_i}$, set

$$\mu^{(k)} = \sum_{i_1, \dots, i_k}^{\neq} \delta_{(x_{i_1}, \dots, x_{i_k})}$$

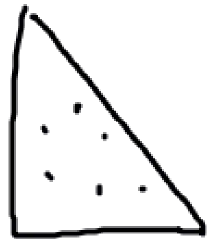
So for $f \in \mathcal{R}_+(\mathbb{X}^k)$

$$\int_{\mathbb{X}^k} f \, d\mu^{(k)} = \sum_{i_1, \dots, i_k}^{\neq} f(x_{i_1}, \dots, x_{i_k})$$

In general, set $\mu(A)$

$$= \int_{\mathbb{X}} \cdots \int_{\mathbb{X}} 1_A(x_1, \dots, x_k) \Big(\mu - \sum_{i=1}^{k-1} \delta_{x_i}\Big)(dx_k)$$

$$\times \cdots \cdot (\mu - \delta_{x_1})(dx_2)\, \mu(dx_1).$$



<u>Theorem</u> (4.4) Let $(\mathbb{X}, \mathcal{X}, \lambda)$ be an s-finite meas. space, η a PPP on $\mathbb{X}$ with mean measure λ and $k \in \mathbb{N}$, $f \in \mathbb{R}_+(\mathbb{X}^k \times \underline{N})$. [write $f(x_1, \dots, x_k; \mu)$ for $f((x_1, \dots, x_k), \mu)$]

$$\mathbb{E}\int_{\mathbb{X}^k} f(x_1,\dots,x_k;\eta)\,\eta^{(k)}(d(x_1,\dots,x_k))$$

$$=\int_{\mathbb{X}}\cdots\int_{\mathbb{X}}\mathbb{E}\Big[f\Big(x_1,\dots,x_k;\eta+\sum_{i=1}^{k}\delta_{x_i}\Big)\Big]\lambda(dx_1)\cdots\lambda(dx_k)$$

e.g. 

$$\mathbb{E}\Big[\sum_{\substack{x,y\in\eta\\ x\neq y}}\sum \mathbb{1}_{\{x,y\ \text{maximal}\}}\Big]$$

$$\Big\| $$

$$\iint_{T\ T} e^{-A(x,y)}\,dx\,dy$$

(η PPP on
Leb = Intens.)

Proof First suppose $\lambda(\mathbb{X}) = \gamma < \infty$

WLOG $\eta = \sum_{i=1}^{\kappa} \xi_i$ where $\kappa \sim \mathrm{Po}(\gamma)$

$\xi_1, \xi_2, \dots \sim \mathbb{Q}$ (indep.) and $\mathbb{Q} = \dfrac{\lambda(\cdot)}{\gamma}$

$$\mathbb{E}\int_{\mathbb{X}^k} f(x_1,\cdots,x_k;\eta)\,\eta^{(k)}(d(x_1\cdots x_k))$$

$$=\sum_{n=k}^{\infty} P[\kappa=n]\,\mathbb{E}[\uparrow \mid \kappa=n]$$

$$=\sum_{n\geq k}\frac{e^{-\gamma}\gamma^n}{n!}\,\mathbb{E}\sum_{i_1,\cdots,i_k\leq n}^{\neq} f(\xi_{i_1},\cdots,\xi_{i_k};\delta_{\xi_1}+\cdots\delta_{\xi_n})$$

$$=\sum_{n=k}^{\infty}\frac{e^{-\gamma}\gamma^n}{\cancel{n!}}\,\frac{\cancel{n!}}{(n-k)!}\int_{\mathbb{X}}\cdots\int_{\mathbb{X}} Q(dx_1)\cdots Q(dx_k)$$

$$\times\,\mathbb{E} f(x_1,\cdots,x_k;\delta_{x_1}+\cdots+\delta_{x_k}+\delta_{\xi_1}+\cdots\delta_{\xi_{n-k}})$$

$$\overset{(m=n-k)}{=}\sum_{m=0}^{\infty}\frac{e^{-\gamma}\gamma^m}{m!}\int\cdots\int \gamma Q(dx_1)\cdots\gamma Q(dx_k)$$

$$\times\,\mathbb{E} f(x_1,\cdots x_k;\delta_{x_1}+\cdots\delta_{x_k}+\delta_{\xi_1}+\cdots\delta_{\xi_m})$$

$$=\int\cdots\int\lambda(dx_1)\cdots\lambda(dx_k)\times\sum_{m=0}^{\infty} P[\eta(\mathbb{X})=m]$$

$$\times\,\mathbb{E}\Big[f(x_1,\cdots,x_k;\eta+\sum_{i=1}^{k}\delta_{x_i})\,\Big|\,\eta(\mathbb{X})=m\Big]$$

$$=\int\cdots\int\lambda(dx_1)\cdots\lambda(dx_k)\,\mathbb{E}\Big[f(x_1,\cdots,x_k;\eta+\sum_{i=1}^{k}\delta_{x_i})\Big].$$

Now suppose $\lambda(\mathbb{X}) = \infty$, just do $k=1$.

Take measures $\lambda_1, \lambda_2, \cdots$ with $\lambda_i(\mathbb{X}) < \infty$

and $\lambda = \sum_{i=1}^{\infty} \lambda_i$. WLOG,

$\eta = \sum_{i=1}^{\infty} \eta_i$ with $\eta_i = PPP(\lambda_i)$ indep

Set $\psi_n = \sum_{i=1}^{n} \eta_i$, $\chi_n = \sum_{i=n+1}^{\infty} \eta_i$. Then

$$\int f(x,\eta)\,\eta(dx) = \sum_{x \in \eta} f(x,\eta) = \lim_{n\to\infty} \sum_{x \in \psi_n} f(x,\eta)$$

$$= \lim_{n\to\infty} \int f(x, \psi_n + \chi_n)\,\psi_n(dx) \quad (*)$$

$$\mathbb{E}\left[\int_{\mathbb{X}} f(x, \Psi_n + \chi_1)\, \Psi_n(dx) \,\middle|\, \Psi_n\right] = g(\Psi_n)$$

indep

where for $\mu \in \underline{N}(\mathbb{X})$ we set

$$g(\mu) := \int \underbrace{\mathbb{E} f(x, \mu + \chi_n)}_{h(x,\mu)} \mu(dx)$$

$$\mathbb{E} \int f(x, \Psi_n + \chi_n)\, \Psi_n(dx) = \mathbb{E}\big[\mathbb{E}[\cdot \mid \Psi_n]\big]$$

$$= \mathbb{E}[g(\Psi_n)] = \int \mathbb{E}\, h(x, \delta_x + \Psi_n)\, \tilde{\lambda}_n(dx)$$

where $\tilde{\lambda}_n = \sum_{i=1}^{n} \lambda_i$ [by earlier case]

$$= \int \mathbb{E}\big[f(x, \delta_x + \underbrace{\Psi_n + \chi_1}_{\eta})\big]\, \tilde{\lambda}_n(dx)$$

By (*) and MON

$$\mathbb{E}\int f(x,\eta)\,\eta(dx)$$

$$= \lim_{n\to\infty} \mathbb{E}\int f(x,\Psi_n+\chi_n)\,\Psi_n(dx$$

$$= \lim_{n\to\infty}\int \mathbb{E}\, f(x,\delta_x+\eta)\,\tilde{\lambda}_n(dx)$$

$$= \int \mathbb{E}\, f(x,\delta_x+\eta)\,\lambda(dx). \quad \blacksquare$$

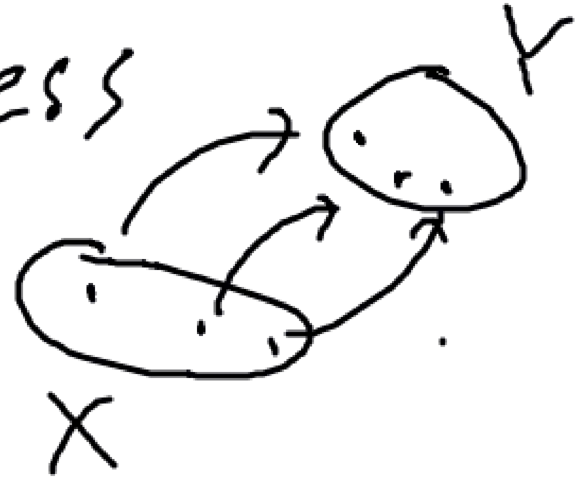
V Mappings, markings & thinnings

V.1 The mapping theorem (5)

Suppose $(\mathbb{X}, \mathcal{X})$ and $(\mathbb{Y}, \mathcal{Y})$ are measurable spaces and $T : \mathbb{X} \to \mathbb{Y}$ is measurable.

Given a proper pt. process

$$\eta = \sum_{i=1}^{\kappa} \delta_{\xi_i} \quad \text{in } \mathbb{X}$$



Set $T(\eta) = \sum_{i=1}^{\kappa} \delta_{T(x_i)}$

(a pt. process in Υ)

$T(\eta)(A) = \eta(T^{-1}(A))$ $\quad A \in \mathcal{Y}$

i.e. $T(\eta) = \eta \circ T^{-1}$ (image / push forward of η)

Use this defn. for any pt. pr. η.

<u>Thm. V.1</u> (5.1). Suppose η is a pt. pr. on $\mathbb{X}$ with mean measure λ. Then

(i) $T(\eta)$ is a pt. pr. in $\mathbb{Y}$ with mean measure $T(\lambda)$

(ii) If also η is a PPP, so is $T(\eta)$.

<u>Proof</u> (i) For $\mu \in \underline{N}_{<\infty}(\mathbb{X})$ and $A_1, A_2, \dots \in \mathcal{Y}$ disjoint

$$T(\mu)(A_i) = \mu(T^{-1}(A_i)) \in \mathbb{N}_0$$

$$T(\mu)\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(T^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right)\right)$$

$$= \mu\left(\bigcup_{i=1}^{\infty} \underbrace{T^{-1}(A_i)}_{\text{disjoint}}\right)$$

$$= \sum_{i=1}^{\infty} \mu(T^{-1}(A_i))$$

and $T(\mu)(\mathbb{Y}) = \mu(\mathbb{X}) < \infty$

so $T(\mu) \in \underline{N}_{<\infty}(\mathbb{Y})$

If $\mu \in \underline{N}(\mathbb{X})$ write

$$\mu = \sum_{i=1}^{\infty} \mu_i, \quad \mu_i \in \underline{N}_{<\infty}(\mathbb{X})$$

$$T(\mu) = \sum_{i=1}^{\infty} T(\mu_i) \in \underline{N}(\mathbb{Y})$$

$T(\eta)$ is a random element of $\underline{N}(\mathbb{Y})$

and for $A \in \mathcal{Y}$, $\mathbb{E}[T(\eta)(A)]$

$$= \mathbb{E}[\eta(T^{-1}(A))] = \lambda(T^{-1}(A))$$

$$= T(\lambda)(A)$$

(ii) if η is a PPP and $A_1, A_2, \cdots A_n \in \mathcal{Y}$ are disjoint then $T^{-1}(A_1), \cdots, T^{-1}(A_n)$ are disjoint so

$$T(\eta)(A_i) = \eta(T^{-1}(A_i)), \quad 1 \leq i \leq n$$

are indep. Poisson. $\blacksquare$