III. 1. Defn. 

\((X, \mathcal{F}, \lambda)\) s-finite 

If \(\mathcal{F}\) is pppl \((\lambda)\) on \(X\): 

- \(\mathcal{F}(A) \sim \mathcal{P}_0(\lambda(A))\) \(\forall A \in \mathcal{F}\) 
- \(\mathcal{F}(A_1) \ldots \mathcal{F}(A_n)\) indep if \(A_i \cap A_j = \emptyset\) if
Theorem III-1 (3.3)
(Superposition)
Suppose \( \lambda_1, \lambda_2, \ldots \) are \( \mathcal{S} \)-finite measures on \((X, \mathcal{X})\) and \( \nu_1, \nu_2, \nu_3, \ldots \) are independent PPPs on \(X\) with mean measures \( \lambda_1, \lambda_2, \ldots \) respectively.
Set \( \eta = \sum_{i=1}^{\infty} \xi_i \), i.e.,

\[ \eta(A) = \sum_{i=1}^{\infty} \xi_i(A), \quad A \in \mathcal{A} \]

Then \( \eta \) is a PPP with mean measure \( \sum_{i=1}^{\infty} \lambda_i \) (which is \( s \)-finite)
Proof: Set \( \xi_n = \sum_{i=1}^{\infty} \xi_i \). Let \( A \in \mathcal{F} \). Then \( \xi_n(A) \sim P_0\left( \sum_{i=1}^{\infty} \lambda_i(A) \right) \) by Prop. I.1 (4 induction).

So for \( k \in \mathbb{N}_0 \),

\[
P[\gamma(A) = k] = \lim_{n \to \infty} P[\xi_n(A) = k]
\]

\[
= \lim_{n \to \infty} P_0\left( \sum_{i=1}^{\infty} \lambda_i(A) > k \right)
\]

\[
= P_0\left( \lambda(A) > k \right), \text{ since } e^{-x} x^k / k! \text{ is cts. in } x.
\]
Also, if $A_1, A_2, \ldots, A_m$ are pairwise disjoint, then

\[ \mathcal{P}_1(A_1), \mathcal{P}_2(A_1), \ldots, \]
\[ \mathcal{P}_1(A_2), \mathcal{P}_2(A_2), \ldots, \]
\[ \mathcal{P}_1(A_m), \mathcal{P}_2(A_m), \ldots, \]

are mutually independent, so

\[ \mathcal{P}(A_1), \mathcal{P}(A_2), \ldots, \mathcal{P}(A_m) \]

\[ \mathcal{P}_1(C A_1), \mathcal{P}_2(C A_1), \ldots, \mathcal{P}_1(C A_m), \mathcal{P}_2(C A_m) \]

are independent.
III.2 Existence of p.p.p.s

Proposition (3.5) Suppose

\((X, \mathcal{F}, \lambda)\) is a meas. space with

\(\lambda(X) < \infty\). There exists a

proper Poisson pt. process on

\(X\) with intensity measure

\(\lambda\).
Proof: On a suitable \((\Omega, \mathcal{F}, \mathbb{P})\) we can arrange to have \(X\) such that

1. A random variable \(K \sim \text{Po}(\lambda)\)
2. A sequence \((\xi_1, \xi_2, \ldots)\) independent random elements of \(X\) with common distribution

\[
Q(\cdot) = \frac{\chi(\cdot)}{\lambda(C(X))} \quad \text{(indep. of } K)\]
Set $\gamma = \sum_{i=1}^{s} \delta_{i}^{\phi} A_{i}$ (mixed binomial rep. of a PPP).

$\gamma$ is a proper Poisson process by def. for $A_1, A_2, \ldots A_n \subset X$. partitioning $X$ setting $p_i = \Omega(A_i)$, $y_i = \gamma(A_i)$, $k \sim Po(\lambda(X))$ and for $k \in \mathbb{N}_0$, $L(y_1, \ldots, y_n | k = k) \sim \text{Mult.}(k; p_1, \ldots, p_n)$ by the extension of Prop. I.2.

$Y_i \sim Po(\lambda(X) \times p_i) = Po(\lambda(A_i))$, and $Y_1, \ldots, Y_n$ are indep.

So $\gamma$ is a PPP $(\lambda)$.
Theorem (3.6) Let $(X, \mathcal{X}, \lambda)$ be an $s$-finite. Then there exists a proper PPP on $X$ with mean measure $\lambda$.

Proof WLOG $\lambda(X) = \infty$. Choose measures $\lambda_1, \lambda_2, \ldots$ on $(X, \mathcal{X})$ with $\lambda_i(X) < \infty$ and $\lambda = \sum_{i=1}^{\infty} \lambda_i$.
On a suitable prob. space, assume we have indep. RVs $K_1, K_2, \ldots$ and $S_{ij}$ (i, j \in \mathbb{N}) with $K_i \sim \mathcal{N}(\lambda_i(X))$ ($\mathbb{N}_0$-valued) and $S_{ij}$ $X$-valued with distribution

$$Q_i = \frac{\lambda_i(X)}{\lambda_i(X)}.$$
For $i \in \mathbb{N}$, set $\mathcal{H}_i = \bigcap_{j=1}^{k_i} \mathcal{S}_{i j}$.

By the lemma, $\mathcal{H}_i$ is a PPP with mean measure $\lambda_i$. Also $\mathcal{H}_1, \mathcal{H}_2, \ldots$ are independent. By Thm. III.1 (Superposition), setting $\mathcal{H} = \bigcup_{i=1}^{\infty} \mathcal{H}_i$, $\mathcal{H}$ is a PPP with mean measure $\sum_{i=1}^{\infty} \lambda_i = \lambda$. Also,
\[ \gamma = \sum_{i=1}^{\infty} \sum_{j=1}^{k_i} s_{ij} \] 

Can rewrite as

\[ \gamma = \sum_{k=1}^{\infty} s_{\Psi_k} \text{, e.g.} \]

\[ k_1 = 2 \]
\[ k_2 = 1 \]
\[ k_3 = 4 \]
\[ k_4 = 3 \]

\[ (\psi_1, \psi_2, \psi_3 \ldots) = (s_{11}, s_{12}, s_{21}, s_{31}, s_{13}, s_{32}, \ldots) \]

so \( \gamma \) is proper. \[ \Box \]
Motivation

Let $\gamma \leq \delta \sum_i x_i$ for all $i \in T$, mean measure $\lambda = \text{Lebesgue}$

Let $Z = \sum_i I_i$

$I_i = \{ \exists l(k(x_i)) = 0 \}$

"$x_i$ maximal"
Find \( E(Z), \ E[Z^2] \)

\[
Z = \sum_{dx \in T} \frac{1}{\mu(dx)} \cdot 1 \times \mathbb{I}_{\eta(K(z))=0} \mathbb{I}_{\eta(K(y))=0}
\]

\[
E(Z) = \int_{T} e^{-\lambda(K(x) \cup K(y))} \ d\lambda
\]

\[
Z^2 - Z = \sum_{i \neq j} I_i I_j
\]

\[
E \sum_{i \neq j} I_i I_j = \int_{T} \int_{T} e^{-\lambda(K(x) \cup K(y))} \ d\lambda \ d\gamma
\]
IV. The univariate Mecke eqn

(4.1) Theorem (4.1). Suppose 

\((X, \mathcal{F}, \lambda)\) is an s-finite measure space and \(\gamma\) is a pt. process on \(X\) with 

\[
\mathbb{E}\int_X f(x, \gamma) \lambda(dx) = \int_X \mathbb{E} f(x, \gamma + \delta_x) \lambda(dx)
\]

\(\forall f \in \mathcal{R}_+(X \times N).\) Then \(\gamma\) is a PPE on \(X\) with mean measure \(\lambda.\)

(\(\diamond\) is the \(\text{univariate Mecke egn}\) \(\) (we'll prove a converse later)
Proof: Let $A_0, \ldots, A_m \in \mathbb{X}$ be disjoint with $\lambda(A_i) < \infty$. Let $k_1, \ldots, k_m \in \mathbb{N}$. Let

$$\left[ f(x, n) \right]_{A_1} \left[ A_2 \right]_{A_2} \left[ A_3 \right]_{A_3}$$
\[ f(x, y) = \mathbb{1}\{x \in A_m, \gamma(A_1) = k_1, \ldots, \gamma(A_m) = k_m\} \]

Then

\[ \mathbb{E} \left[ \sum \mathbb{1}\{\gamma(A_i) = k_i\} \prod_{i=1}^{m-1} \mathbb{1}\{\gamma(A_i) = k_i\} \right] \]

\[ = k_m \mathbb{P}[\gamma(A_1) = k_1, \ldots, \gamma(A_m) = k_m] \]

and for \( x \in X \)

\[ \mathbb{E} f(x, \delta_x + y) = \mathbb{1}_{A_m} \mathbb{P}[\gamma(A_1) = k_1, \gamma(A_2) = k_2, \ldots, \gamma(A_m) = k_{m-1}] \]

So by (x)
\[ k_m P \left[ \gamma(A_1) = k_1, \ldots, \gamma(A_m) = k_m \right] = \lambda(A_m) P \left[ \bigcap_{i=1}^{m-1} \{ \gamma(A_i) = k_i \} \land \{ \gamma(A_m) = k_m \} \right] \]

Set \( \Pi(k) = P(\gamma(A_m) = k) \left[ \bigcap_{i=1}^{m-1} \{ \gamma(A_i) = k_i \} \right] \)

\[ \Pi(k) = \frac{\lambda(A_m)}{k_m} \Pi(k-1) \]

(\( k = k_m \))

\[ \Pi(n) = \Pi(0) \frac{\Pi(1)}{\Pi(0)} \ldots \frac{\Pi(n)}{\Pi(n-1)} = \frac{\Pi(0) \lambda(A_m)}{n!} \]

So \( \Pi(n) \) is \( Po(\lambda(A_m); n) \) does not depend on \( k_1, \ldots, k_{m-1} \)

\( \gamma(A_m) \) is \( Po(\lambda(A_m)) \) independent of \( \gamma(A_1), \ldots, \gamma(A_{m-1}) \)

\( \gamma(A_i) = k_i \)
By induction on $m$, $\gamma(A_1), \ldots, \gamma(A_m)$ are indep.

If $\lambda(A) = \infty$ then still

$$k \mathbb{P}[\gamma(A) = k] = \lambda(A) \mathbb{P}[\gamma(A) = k - 1]$$

So $\mathbb{P}[\gamma(A) = k - 1] = 0 \quad \forall k \in \mathbb{N}$

So $\mathbb{P}[\gamma(A) = \infty] = 1$

So $\gamma$ is a PPP with mean measure $\lambda$. \qed