II.3 Distribution of pt. processes

(2.3)

Given random elements ξ, ξ' of a measurable space $(Y, \mathcal{Y})$ we write $\xi \stackrel{d}{=} \xi'$ if the measure $\mathbb{P}_\xi$ on $(Y, \mathcal{Y})$ given by $\mathbb{P}_\xi(\cdot) = \mathbb{P}[\xi \in \cdot]$ coincides with $\mathbb{P}_{\xi'}$.

We call $\mathbb{P}_\xi$ the distribution of ξ. For $f \in \mathbb{R}_+(\mathbb{Y})$

$$\mathbb{E}[f(\xi)] = \int f \, d\mathbb{P}_\xi,$$

The distribution of a pt. process η on $(\mathbb{X}, \mathcal{X})$ is the measure $\mathbb{P}_\eta$ on $(\underline{N}, \mathcal{N})$ given by

integer-valued measures on $\mathbb{X}$

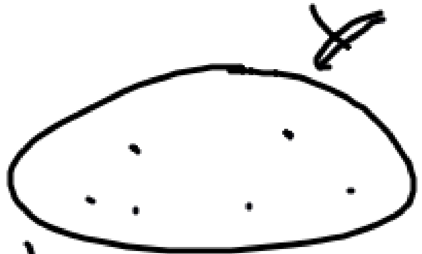
$$\mathbb{P}_\eta(\cdot) = \mathbb{P}[\eta \in \cdot]$$

Definition (2-8) Given a pt. process η on $(\mathbb{X}, \mathcal{X})$ the **Laplace functional** of η is the mapping $L_\eta : \mathbb{R}_+(\mathbb{X}) \to [0,1]$

$$L_\eta(u) = \mathbb{E}\, e^{-\eta(u)} \quad \text{for } u \in \mathbb{R}_+(\mathbb{X})$$

where $\lambda(u) := \int u \, d\lambda$ and λ a measure on $(\mathbb{X}, \mathcal{X})$

Example Let η be the binomial pt. process of example II.2. Then



$$L_\eta(u) = \mathbb{E}\left[\exp\left(-\sum_{i=1}^m u(X_i)\right)\right]$$

[where $X_1 \dots X_m$ iid $\sim Q$ on $\mathcal{X}$]

$$= \mathbb{E}\left[\prod_{i=1}^m e^{-u(X_i)}\right]$$

$$= \prod_{i=1}^m \mathbb{E}\left[e^{-u(X_i)}\right] = \left(\int_{\mathcal{X}} e^{-u(x)}\, Q(dx)\right)^m$$

Proposition (2.10) For point processes η, η' on $(\mathbb{X}, \mathcal{X})$, the following are equivalent

(i) $\eta \overset{d}{=} \eta'$

(ii) $(\eta(B_1), \eta(B_2), \cdots, \eta(B_m)) \overset{d}{=} (\eta'(B_1), \cdots, \eta'(B_m))$ $\forall m \in \mathbb{N}$, $B_1, \cdots, B_m \in \mathcal{X}$ pairwise disjoint

(iii) $L_\eta(u) = L_{\eta'}(u)$ $\forall u \in \mathbb{R}_+(\mathbb{X})$

(iv) $\forall u \in \mathbb{R}_+(\mathbb{X})$, $\eta(u) \overset{d}{=} \eta'(u)$

In partic., the Laplace functional characterizes the distribution of η.

<u>Proof</u> (i) $\Longrightarrow$ (iv). Given $u \in R_+(\mathbb{X})$

$g_u : \xi \longrightarrow \int_{\mathbb{X}} u \, d\xi$ is measurable

$\underline{N} \to \mathbb{R}$, so

$$\mathbb{P}_{\eta(u)}(\cdot) = \mathbb{P}[\eta(u) \in \cdot]$$

$$= \mathbb{P}[g_u(\eta) \in \cdot] = \mathbb{P}[g_u(\eta') \in \cdot]$$

(if we assume (i))

$$= \mathbb{P}_{\eta'(u)}(\cdot) \quad \text{so } \eta(u) \overset{d}{=} \eta'(u)$$

(iv) $\Longrightarrow$ (iii). If $\eta(u) \stackrel{d}{=} \eta'(u)$ $\forall u \in \mathcal{R}_+(\mathbb{X})$

then $L_\eta(u) = \mathbb{E}\, e^{-\eta(u)}$

$$= \int e^{-x}\, \mathbb{P}_{\eta(u)}(dx)$$

$$= \int e^{-x}\, \mathbb{P}_{\eta'}(dx) = L_{\eta'}(u)$$

$\forall u \in \mathcal{R}_+(\mathbb{X})$.

(iii) $\Rightarrow$ (ii) Assume $L_\eta(u) = L_{\eta'}(u)$ $\forall u$.

Let $B_1, \dots, B_m \in \mathcal{X}$ (pw disjoint)

Set $u(x) = \sum_{i=1}^m a_i \mathbb{1}_{B_i}(x)$

with $a_i > 0$. Then

$$L_\eta(u) = \mathbb{E} \exp\Big(-\sum_{i=1}^m a_i \underbrace{\eta(B_i)}_{\xi_i}\Big)$$

$$= \hat{\mathbb{P}}_{\xi_1, \dots, \xi_m}(a_1, \dots a_m)$$

where $\hat{\mathbb{P}}_{\xi_1, \dots \xi_m}$ is the multivariate Laplace transform (MVLT) of the distribution of $(\xi_1, \dots \xi_m)$

given measure μ on $[0,\infty]^m$

define MVLT

$$\hat{\mu}(a_1 \cdots a_m) = \int_{[0,\infty]^m} e^{-a \cdot x} \mu(dx)$$

$(a_i > 0$

$\hat{\mu}$ determines μ so

$$\mathbb{P}_{\xi_1, \ldots, \xi_m} = \mathbb{P}_{\xi'_1, \ldots, \xi'_m}$$

$[\xi'_i = \eta'(B_i)]$

i.e. (ii)

(ii) $\Rightarrow$ (i). Let Π be the collection of all subsets of $\underline{N}$ of the form

$$V = \{\mu \in \underline{N} : \mu(A_1) = k_1, \dots \mu(A_m) = k_m\}$$

with $m \in \mathbb{N}$, $A_1, \cdots, A_m \in \mathcal{H}$, $k_1, \dots k_m \in \mathbb{N}_0$

Π is a π-system generating $\mathcal{N}$

For $\alpha \subset \{1, \cdots, m\}$, let $B_\alpha = \bigcap_{i \in \alpha} A_i \cap \bigcap_{i \notin \alpha} A_i^c$

$\{B_\alpha : \alpha \subset \underbrace{\{1, \cdots m\}}_{[m]}\}$ are disjoint

$$A_i = \bigcup_{\alpha \subset [m] : i \in \alpha} B_\alpha$$

$$V = \{\mu \in \underline{N} : \mu(A_1) = k_1, \dots \mu(A_m) = k_m\}$$

$$\mathbb{P}_\eta(V) = \mathbb{P}\Big[\sum_{\alpha \ni 1} \eta(B_\alpha) = k_1, \dots \sum_{\alpha \ni m} \eta(B_\alpha) = k_m\Big]$$

$$\overset{\text{by (ii)}}{=\!=} \mathbb{P}[\ \text{---}\ \eta'\ \text{------}\ \eta'\ \text{--}\ ]$$

$$= \mathbb{P}_{\eta'}(V)$$

So by Uniqueness Lemma,

$$\mathbb{P}_\eta(W) = \mathbb{P}_{\eta'}(W) \quad \forall W \in \sigma(\Pi) = N$$

i.e. (i). $\blacksquare$

II.4 Point processes on metric spaces (2.4)

In this section, assume $(\mathbb{X}, \rho)$ is a metric space and $\mathcal{X}$ is the Borel σ-field on $\mathbb{X}$.

We say $B \subset \mathbb{X}$ is <u>bounded</u> if $B = \emptyset$ or $\operatorname{diam}(B) := \sup\{\rho(x,y) : x, y \in B\} < \infty$.

Defn (2-11, 2-13). We say a measure μ on $\mathbb{X}$ is locally finite if $\mu(B) < \infty \ \forall B \in \mathcal{X}_b := \{\text{bounded sets in } \mathbb{X}\}$

We say a pt. process η on $\mathbb{X}$ is locally finite if

$$\mathbb{P}[\eta(B) < \infty] = 1 \quad \forall B \in \mathcal{X}_b$$

Proposition (2.12). Suppose η, η' are pt. processes in $\mathbb{X}$ with $\eta(u) \overset{d}{=} \eta'(u)$ $\forall\, u \in \mathcal{R}_+(\mathbb{X})$ with

$$\operatorname{diam}(\{x : u(x) \neq 0\}) < \infty \quad (*)$$

Then $\eta \overset{d}{=} \eta'$.

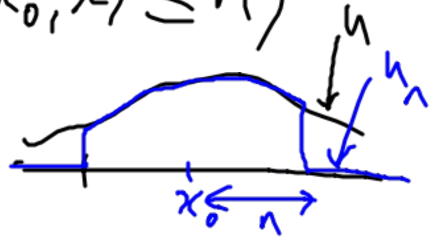
Proof Let $u \in \mathcal{R}_+(\mathbb{X})$.

<u>Proof</u> Let $u \in \mathbb{R}_+(\mathbb{X})$

Fix $x_0 \in \mathbb{X}$ and set

$$u_n(x) = u(x)\, \mathbb{I}\{d(x_0, x) \le n\}$$

Then $u_n \uparrow u$ and



(*) holds for u_n

By MON $\eta(u_n) \uparrow \eta(u)$ as. $n \to \infty$, a.s.

By our assumption and DOM

as $n \to \infty$ (a.s.

$$L_\eta(u) = \mathbb{E}e^{-\eta(u)}$$

$$= \lim_{n\to\infty} \mathbb{E}e^{-\eta(u_n)} \quad \text{(Dom)}$$

$$\eta(u_n) \to \eta(u)$$

$$= \lim_{n\to\infty} \mathbb{E}e^{-\eta'(u_n)} = L_{\eta'}(u)$$

So $\eta \overset{d}{=} \eta$ by Proposition II.3

III Poisson Processes (3)

III.1 The definition (3.1)

Given an s-finite measure λ
on a measurable space $(\mathbb{X}, \mathcal{X})$
a Poisson (point) process (PPP)
on $\mathbb{X}$ with intensity measure
λ is a point process on $\mathbb{X}$ with:

(i) $\eta(A) \sim Po(\lambda(A)) \quad \forall A \in \mathcal{X}$

(ii) For $A_1, A_2, \cdots A_n \in \mathcal{X}$ pw disjoint:

$\eta(A_1), \cdots, \eta(A_n)$ are indep.

[i.e. η is completely independent]

By Prop. II.3, if η' also satisfies (i) & (ii), then $\eta \overset{d}{=} \eta'$

Ex. (a) Give an example with (i) but not (ii)

(b) Show if (ii) holds only for $n=2$ it might not be a PPP