Poisson Processes & Point Patterns

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Book: Last & P.

Chapters 1–5, 8, 9, 12, ?

I The Poisson distribution

(1 in book)

Let $\gamma \geq 0$. A random variable X is Poisson with parameter γ if X takes values in $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ and for $k \in \mathbb{N}_0$,

$P[X = k] = P_0(\gamma; k)$, where

$$P_0(\gamma; k) = \begin{cases} e^{-\gamma} \gamma^k / k! & \gamma > 0 \\ 1_0(k) & \gamma = 0 \end{cases}$$

Allow $\gamma = \infty$: then take

$P[X = \infty] = 1$

Suppose $X \sim Po(\gamma)$ $(0<\gamma<\infty)$

For $s \geq 0$, $t \geq 0$

$$\mathbb{E}[s^X] = \sum_{k=0}^{\infty} s^k e^{-\gamma} \gamma^k / k!$$

$$= e^{-\gamma} e^{s\gamma} = e^{\gamma(s-1)} \quad \text{(MGF)}$$

$$\mathbb{E}[e^{-tX}] = e^{\gamma(e^{-t}-1)} \quad \text{(Laplace Trans)}$$

Also $\mathbb{E}[X] = \gamma$

$\mathbb{E}[(X)_k] = \gamma^k$ where

$$(n)_k := n(n-1)\cdots(n-k+1)$$

Prop. I.1 (1.2). Let

$\gamma, \delta \geq 0$ with $\gamma + \delta > 0$.

Suppose X, Y are indep. with $X \sim Po(\gamma)$, $Y \sim Po(\delta)$.

(a) $X + Y \sim Po(\gamma + \delta)$

(b) $\mathcal{L}(X \mid X+Y = k) = Bin\left(k, \frac{\gamma}{\gamma+\delta}\right)$

Proof for $j, k \in \mathbb{N}_0$

$$P[X=j, X+Y=k] = \frac{e^{-\gamma}\gamma^j}{j!}\,\frac{e^{-\delta}\delta^{k-j}}{(k-j)!}$$

$$= \frac{e^{-(\gamma+\delta)}(\gamma+\delta)^k}{k!}\binom{k}{j}\left(\frac{\gamma}{\gamma+\delta}\right)^j\left(\frac{\delta}{\gamma+\delta}\right)^{k-j}$$

Sum over j to get (a)

Then get (b)

Prop. I.2 (1.3) (Poisson thinning)

Let $\lambda > 0$, $p \in [0,1)$. If $Z \sim Po(\lambda)$ and $\forall k \in \mathbb{N}_0$,

$$\mathcal{L}(X \mid Z = k] = Bin(k, p)$$

Then $X \sim Po(\lambda p)$ and X is indep. of $Z - X$.

Proof For $j, k \in \mathbb{N}_0$,

$$P[X = j, Z - X = k]$$

$$= P[Z = k+j]\, P[X = j \mid Z = k+j]$$

$$= \frac{e^{-\lambda} \lambda^{k+j}}{(k+j)!} \binom{j+k}{j} p^j (1-p)^k$$

$$= P_0(\lambda p; j)\, P_0(\lambda(1-p); k) \quad \blacksquare$$

Extension (Poisson colouring)

If $Z \sim Po(\lambda)$ & for some $p_1, p_2, \cdots p_m \geq 0$ with $\sum_{i=1}^{m} p_i = 1$,

$\mathcal{L}\left((X_1, \cdots, X_m) \,\middle|\, Z = k \right] = \text{mult}\left(k, p_1, \cdots, p_m\right)$

$\forall k \in \mathbb{N}_0$. Then

$X_1, \cdots, X_m$ are indep.

with $X_i \sim Po(\lambda p_i)$, $1 \leq i \leq m$.

II. Point Processes

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space, i.e. $\mathbb{X}$ is a set (e.g. $\mathbb{R}^n$) and $\mathcal{X}$ is a σ-field ($=\sigma$-algebra) of subsets of $\mathbb{X}$.

Aim: model a random set of points in $\mathbb{X}$.

Could consider a set of $\mathbb{X}$-valued RVs $\{\xi_1, \xi_2, \xi_3, \dots\}$

Don't care about order: may have multiplicities.

We'll represent this set as a point measure η on $(\mathbb{X}, \mathcal{X})$ where
$\eta(A) = \#$ points in A
where $A \in \mathcal{X}$, $\eta(A) \in \overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$

Given $(\mathbb{X}, \mathcal{X})$, let $\underline{N}_{<\infty} = \underline{N}_{<\infty}(\mathbb{X})$ be the class of all finite integer-valued measures on $(\mathbb{X}, \mathcal{X})$. Let $\underline{N}$ be the class of all measures that can be written as a countable sum of measures in $\underline{N}_{<\infty}$

ʃ.

Examples (a) Given $x \in \mathbb{X}$ the Dirac measure δ_x is:

$$\delta_x(B) = \mathbb{1}_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if not} \end{cases}$$

(b) Let $k \in \overline{\mathbb{N}}_0$ and let $(x_n)_{n=1}^{k}$ be elements of $\mathbb{X}$. Then

$$\sum_{i=1}^{k} \delta_{x_i} \in \underline{N}\ldots$$

We say a measure ν on $\mathbb{X}$ is s-finite if it is a countable sum of finite measures.

σ-finite $\Rightarrow$ s-finite

σ-finite $\Rightarrow$ s-finite

Every measure in $\underline{N}$ is s-finite.

Define the σ-field $\mathcal{N}$ on $\underline{N}$:

$$\mathcal{N} = \sigma(\{\mu \in \underline{N} : \mu(B) = k\} : B \in \mathcal{X}, k \in \mathbb{N}_0)$$

Defn. II.1 (2.1):

A <u>point process</u> on $\mathbb{X}$ is a measurable function

$$\eta : \Omega \longrightarrow \underline{N}$$

[for some prob. space $(\Omega, \mathcal{F}, P)$]

Often write η for $\eta(\omega)$, $\omega \in \Omega$

$\eta(B)$ for $\eta(\omega)(B)$ $(B \in \mathcal{X})$.

$\eta(B)$ is a R.V.

Example II.2 (2-3)

Suppose $\mathbb{Q}$ is a prob. measure on $(\mathbb{X}, \mathcal{X})$ and $\xi_1, \xi_2, \ldots$ are i.i.d. $\mathbb{Q}$-distributed random elements of $\mathbb{X}$ [on $(\Omega, \mathcal{F}, P)$]. Let $n \in \mathbb{N}$.

Set $\eta = \sum_{i=1}^{n} \delta_{\xi_i}$.

Then η is a point process on $\mathbb{X}$. $\eta(A) = \#$ points $\xi_i \in A$ (for $A \in \mathcal{X}$). To check measurability

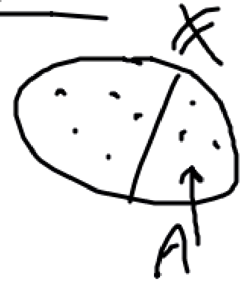
for $n=1$:

$$\{\omega : \eta(A) = 1\} = \{\omega : \xi_1(\omega) \in A\} \in \mathcal{F}$$

[so for general n: sum of meas. fns.]

η is called a <u>binomial</u> point process since

$$\eta(A) \sim \mathrm{Bin}(n, Q(A))$$

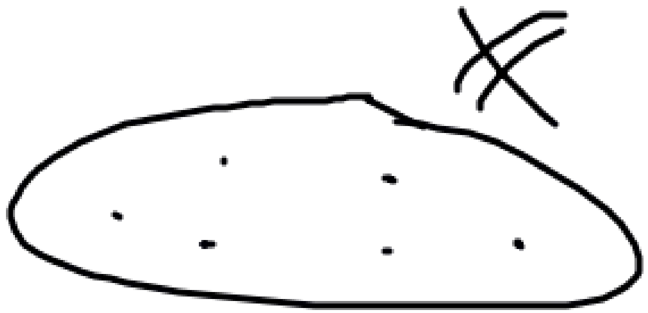


We say pt. processes η and η' on $\mathbb{X}$ are al*most surely equal* if $\exists\, A \in \mathcal{F}$ with $P[A]=1$ and $\eta(\omega)=\eta'(\omega)\ \forall \omega \in A$.

Defn. II.4 (2.3) We say a pt. process η is a *proper point process* if

if $\exists$ random elements $X_1, X_2, \ldots \in \mathbb{X}$ and a $\mathbb{N}_0$-valued R.V. K with

$$\eta = \sum_{n=1}^{K} \delta_{X_n} \quad \text{a.s.}$$



II.3 Campbell's formula

Given a point process η on $(\mathbb{X}, \mathcal{X})$, its **intensity measure** is the measure λ on $(\mathbb{X}, \mathcal{X})$:

$$\lambda(A) = \mathbb{E}[\eta(A)]$$

Exercises: (a) λ is a measure
(b) For example II.2, $\lambda(A) = n\mathbb{Q}(A)$

Notation: Given $(X, \mathcal{X})$

$\mathbb{R}(X) = \{\mathcal{X}\text{-measurable fns } X \to \mathbb{R}\}$

$\overline{\mathbb{R}}(X) = \{ \text{---------} \quad X \to [-\infty, \infty] \}$

$\mathbb{R}_+(X) = \{ \text{---------} \quad X \to [0, \infty) \}$

$\overline{\mathbb{R}_+}(X) = \{ \text{---------} \quad X \to [0, \infty] \}$

For any measure μ on $(X, \mathcal{X})$ and $u \in \overline{\mathcal{R}}(X)$, define

$$\int u \, d\mu = \int u^+ \, d\mu - \int u^- \, d\mu$$

with the convention $\infty - \infty = 0$

$[u^+(x) = u(x) \vee 0, \; u^-(x) = -u(x) \vee 0]$

Proposition II.5 (2.7) (Campbell's Formula)

If η is a pt. process on $(\mathbb{X}, \mathcal{X})$ with intensity λ and $u \in \overline{\mathbb{R}}(\mathbb{X})$ then $\int_{\mathbb{X}} u \, d\eta$ is a R.V. and

$$\mathbb{E}\left[\int_{\mathbb{X}} u \, d\eta\right] = \int_{\mathbb{X}} u \, d\lambda \qquad (*)$$

provided either $u \geq 0$ or $\int_{\mathbb{X}} |u| \, d\lambda < \infty$

Proof Suppose $u = \mathbb{1}_A$ with $A \in \mathcal{X}$. Then

$$\int_{\mathcal{X}} u \, d\eta = \eta(A), \text{ a R.V.}$$

$$\mathbb{E} \int_{\mathcal{X}} u \, d\eta = \lambda(A) = \int u \, d\lambda \text{ so } (*)$$

Extend to simple u by linearity, then to non-negative u by MON, and to general u by considering u^+ & u^-