MA40092 PROBLEM SHEET 8

Bookwork question (§5.1)
Suppose that we are interested in testing the hypotheses

$$H_0 : \theta \in \Theta_0 \quad \text{vs} \quad H_1 : \theta \in \Theta_1$$

Define the size and power of such a test.
Define what is meant by a Uniformly Most Powerful Test (UMP).
Define what is meant by saying that a joint distribution $f_\theta(\mathbf{x})$ has a Monotone Likelihood Ratio in a statistic $T(\mathbf{x})$.
State and prove a theorem which allows you to construct a uniformly most powerful test of the hypotheses

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0$$

when $f_\theta(\mathbf{x})$ has a Monotone Likelihood Ratio in a statistic $T(\mathbf{x})$.

Example 2: Monotone likelihood ratio tests (§5.1)
Suppose random variable X has density function $f_\theta(x) = \theta x^{-(\theta+1)}$ for $x \geq 1$ and unknown $\theta > 0$. Show that the size α uniformly most powerful test of

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0$$

based on the random variable X is given by

$$\text{Reject } H_0 \Leftrightarrow X \leq (1-\alpha)^{-(1/\theta_0)}.$$

Example 3: Generalised likelihood ratio tests (§5.2)
Suppose $X_1, \ldots, X_n$ are iid random variables following a double exponential distribution, that is each has probability density function

$$f_{\eta,\mu}(x) = \frac{1}{2\mu} \exp(-|x-\eta|/\mu), \quad x \in (-\infty, \infty)$$

where η is a constant and μ is a positive constant.
Find the maximum likelihood estimators of μ and η and construct the Generalised Likelihood Ratio test of the pair of hypotheses

$$H_0 : \eta = 0 \quad \text{versus} \quad H_1 : \eta \neq 0$$

when the parameter μ is unknown.

Example 4: Generalised likelihood ratio tests (§5.2)
Suppose $X_1, \ldots, X_n$ are independent normal random variables with distributions $X_i \sim N(a+b\gamma_i, \sigma^2)$ where the γ_i are known constants satisfying $\sum_{i=1}^n \gamma_i = 0$, and σ^2 is also known, while a and b are unknown parameters and n is large. Find the Generalised Likelihood Ratio Test of

$$H_0 : b = 0 \qquad \text{versus} \qquad H_1 : b \neq 0$$

Under the null hypothesis what is the approximate (asymptotic for large n) distribution of your test statistic?