**MA40092 PROBLEM SHEET 7**

**Example 1: Neyman-Pearson lemma, UMP tests** (§4.1, §4.2)
A single positive random variable $X$ has density function

$$f_\theta(x) = \frac{\theta}{(\theta + x)^2}, \qquad x > 0,$$

where $\theta > 0$ is an unknown parameter. Find the form of the optimal test of

$$H_0 : \theta = 2 \qquad \text{versus} \qquad H_1 : \theta > 2.$$

Find the Uniformly Most Powerful test of size 0.05 for this pair of hypotheses.

**Example 2: Neyman-Pearson lemma, UMP tests** (§4.1, §4.2)
Suppose we have a single observation from a $N(\theta, \theta^2)$ distribution. Derive the form of the critical region of the best test of

$$H_0 : \theta = 1 \qquad \text{versus} \qquad H_1 : \theta = \theta_1$$

where $\theta_1$ is a fixed constant greater than 1.
How does this example show that there are most powerful tests which are not uniformly most powerful?

**Example 3: The Neyman-Pearson lemma** (§4.1)
Prove that for a test of a pair of simple hypotheses derived using the Neyman-Pearson Lemma, the power is at least as large as the size of the test.
[Hint: Express the size of the test in terms of $\int_{C^*} f_0(\mathbf{x})d\mathbf{x}$ and then in terms of $\int_{\bar{C}^*} f_0(\mathbf{x})d\mathbf{x}$.]

**Example 4: Monotone likelihood ratio tests** (§5.1)
Suppose that $X$ has the Weibull distribution with density

$$f_{\theta,\gamma}(x) = \theta\gamma x^{\gamma-1} \exp\{-\theta x^\gamma\}, \qquad x > 0,$$

for $\theta > 0$ unknown, and $\gamma > 0$ a known parameter. Given $X_1, \ldots, X_n$ drawn independently from this distribution, find a statistic $T(\mathbf{X})$ such that $f_{\theta,\gamma}(x)$ has the monotone likelihood ratio property with respect to $T$. What is the form of the UMP test of

$$H_0 : \theta \leq \theta_0 \qquad \text{versus} \qquad H_1 : \theta > \theta_0?$$