MA40092 PROBLEM SHEET 6

Bookwork question: (§1.4)
Prove (in the case that $g()$ is an invertible function) that maximum likelihood estimates are functionally invariant, that is if $\hat{\theta}$ is the MLE of θ, then the MLE of $g(\theta)$ is $g(\hat{\theta})$.

Example 2: Maximum likelihood estimation (§1.4, §3.1)
A random sample of n observations $X_1, \ldots, X_n$ is taken from a distribution with probability mass function

$$\begin{aligned} \mathsf{P}(X=0) &= \theta_1\theta_2 \\ \mathsf{P}(X=1) &= \theta_1(1-\theta_2) \\ \mathsf{P}(X=2) &= (1-\theta_1)\theta_2 \\ \mathsf{P}(X=3) &= (1-\theta_1)(1-\theta_2) \end{aligned}$$

with $0 < \theta_1 < 1$, and $0 < \theta_2 < 1$. If n_0, n_1, n_2 and n_3 are the observed numbers of 0's, 1's, 2's and 3's, find the maximum likelihood estimators of θ_1 and θ_2 along with their asymptotic joint distribution.

Example 3: Maximum likelihood estimation (§1.4, §2.3, §3.1)
Suppose $X_1, \ldots, X_n$ are independent Poisson random variables with mean θ. Let $\phi = \sqrt{\theta}$. Find the maximum likelihood estimator of ϕ and show that its asymptotic variance is independent of θ.
What is the connection between $I_n(\theta)$ and $I_n(\phi)$?

Example 4: Maximum likelihood estimation (§1.4, §2.3, §3.1, §3.2)
Suppose $X_1, \ldots, X_n$ are independently and identically distributed with distribution

$$f_{\lambda,\eta}(x) = \lambda \exp\left(-\lambda(x-\eta)\right), \ x > \eta$$

where λ and η are positive parameters with η known but λ unknown. Find the MLE of λ and construct a $(1-\alpha)100\%$ confidence interval for λ when n is assumed to be large.