MA40092 PROBLEM SHEET 5

Bookwork question (§2.3)
Carefully state and prove the Cramer-Rao Lower Bound for the variance of an unbiased estimator $T(\mathbf{X})$ of a function of a parameter, $\phi(\theta)$, based on $X_1, X_2, \ldots, X_n$ which have joint distribution $f_\theta(\mathbf{x})$. Further show that:

- $I_n(\theta)$, Fisher's information based on a sample of size n, can also be written as a function of the second derivative of the log likelihood
$$\mathsf{E}\left[\left(\frac{d \ln f_\theta(\mathbf{x})}{d\theta}\right)^2\right] = -\mathsf{E}\left[\frac{d^2 \ln f_\theta(\mathbf{x})}{d\theta^2}\right]$$
- In the case where $X_1, X_2, \ldots, X_n$ are independently and identically distributed each with marginal distribution $f_\theta(x)$
$$I_n(\theta) = nI_1(\theta)$$
- Show that when the Cramer-Rao Lower Bound for estimating $\phi(\theta)$ is attained by an unbiased estimator $T(\mathbf{X})$, it is possible to write
$$\frac{d \ln f_\theta(\mathbf{x})}{d\theta} = a(\theta)\{T(\mathbf{x}) - \phi(\theta)\}$$
for some $a(\theta)$. Show that in this case, $T(\mathbf{X})$ is a sufficient statistic.

Example 2: Cramer-Rao, MVUE (§2.3, §2.4)
Suppose $X_1, \ldots, X_n$ are independent, identically distributed $Poi(\mu)$ random variables. Calculate the Cramer-Rao Lower Bound for estimating $e^{-\mu}$. Does an unbiased estimator exist which achieves full efficiency?

Example 3: Cramer-Rao (§2.3)
Explain the apparent contradiction in the following: If $X_1, \ldots, X_n$ are independent, identically distributed $U(0, \theta)$ random variables, then an unbiased estimator for θ is $\hat{\theta} = \frac{n+1}{n} \max_i X_i$, which has
$$\mathrm{Var}(\hat{\theta}) = \frac{\theta^2}{n(n+2)} \text{and } \mathsf{E}\left[\left(\frac{d \ln f_\theta(\mathbf{x})}{d\theta}\right)^2\right] = \frac{n^2}{\theta^2}$$

Example 4: Cramer-Rao, MVUE, efficiency (§2.3, §2.4)
Derive the Cramer-Rao lower bound for the variance of an unbiased estimator based on an independent sample of size n from a distribution with pdf
$$f_\theta(x) = \frac{3\theta^3}{(x+\theta)^4}, \qquad x > 0, \ \theta > 0$$
Does an estimator which attains this bound exist? Show that $2\bar{X}$ is an unbiased estimator for θ and find its efficiency.