MA40092 PROBLEM SHEET 4

Bookwork question (§2.2)
State and prove the Rao-Blackwell theorem.

Example 2: Cramer-Rao (§2.3)
Suppose we have an independent sample of size n from the Poisson distribution with mean λ. Find the Cramer-Rao Lower Bound for estimating λ. Find the variance of $Y = \sum_{i=1}^{n} X_i$ and show that Y/n attains the CRLB.

Example 3: Cramer-Rao (§2.3)
Let $X_1, \ldots, X_n$ be independent, identically distributed random variables with density

$$f_{\alpha,l}(x) = \frac{2l^{\alpha} x^{-(2\alpha+1)} e^{-l/x^2}}{\Gamma(\alpha)}, \qquad x > 0,$$

for a known parameter $\alpha > 0$, but an unknown parameter $l > 0$. Find the Cramer-Rao Lower Bound on the variance of an unbiased estimator of l.

Example 4: Cramer-Rao, efficiency (§2.3)
Suppose $X_1, \ldots, X_n$ are independent and identically distributed with density $f_\theta(x) = \theta^{-1/2} e^{-x/\sqrt{\theta}}$, for $x > 0$. Define $T = Z^2/(n(n+1))$ where $Z = \sum_{i=1}^{n} X_i$. Using the fact that

$$\mathsf{E}(Z^k) = \frac{(n+k-1)!\theta^{k/2}}{(n-1)!}$$

show that T is an unbiased estimator of θ, and calculate its variance.
Find the Cramer-Rao Lower Bound for the variance of an unbiased estimator of θ, and show that the efficiency of T relative to this bound is $2(n+1)/(2n+3)$.