MA40092 PROBLEM SHEET 3

Example 1: Sufficiency, bias, Rao-Blackwell theorem (§1.3, §2.1, §2.2)
Suppose $X_1, \dots, X_n$ are independent $exp(\lambda)$ random variables, where λ is the rate parameter. Find a sufficient statistic for the distribution.
Find an unbiased estimator for the mean of the $exp(\lambda)$ distribution based solely on X_1, and apply the Rao-Blackwell theorem to improve your estimator.
Compare the variances of your initial estimator and your improved estimator.

Example 2: Sufficiency, bias, Rao-Blackwell theorem (§1.3, §2.1, §2.2)
Suppose $X_1, \dots, X_n$ are independent Poisson random variables with mean λ.
Show that $Y = \sum_{i=1}^{n} X_i$ is a sufficient statistic for λ.
Find an unbiased estimator of $p_r = \mathsf{P}(X = r)$, which depends only upon X_1.
Find $\mathsf{P}(X_1 = r | Y = k)$ both for $k \geq r$ and $k < r$. Hence use the Rao-Blackwell Theorem to improve your estimator of p_r.

Example 3: Sufficiency, bias, Rao-Blackwell theorem (§1.3, §2.1, §2.2)
Let $X_1, \dots, X_n$ be independent Bernoulli random variables with mean p. Write down a sufficient statistic for p.
Find an unbiased estimator of $\mathrm{Var}(X) = p(1-p)$ using only X_1 and X_2.
Now apply the Rao-Blackwell Theorem to obtain a better unbiased estimator of $p(1-p)$.

Example 4: Sufficiency, bias, Rao-Blackwell theorem (§1.3, §2.1, §2.2)
Suppose X_1 and X_2 are independent $N(\mu, 1)$ random variables, for an unknown parameter μ. Find a sufficient statistic for μ.
Find a function of X_1 which is an unbiased estimator of μ^2.
Apply the Rao-Blackwell theorem to improve your initial estimator.
[Hint: you will need to use the relation $Var(X) = \mathsf{E}(X^2) - (\mathsf{E}(X))^2$ and to show that $X_1 | (X_1 + X_2 = s) \sim N(s/2, 1/2)$.]