Sketch proof for univariate θ

Given that we are discussing ML under suitable regularity conditions,

$$\frac{d\ell}{d\theta}(\hat{\theta}) = 0 \qquad (\ell'(\hat{\theta}) = 0)$$

ie not at boundary

(Now further assume $X_1 \ldots X_n$ iid say later when needed)

Taylor-expand ℓ' about the "true" value θ, so that

$$0 = \ell'(\hat{\theta}) \approx \ell'(\theta) + (\hat{\theta} - \theta)\ell''(\theta)$$

$$\Rightarrow \quad (\hat{\theta} - \theta) \approx \frac{-\ell'(\theta)}{\ell''(\theta)}$$

$$\sqrt{n}(\hat{\theta} - \theta) \approx \frac{-1/\sqrt{n}\,\ell'(\theta)}{1/n\,\ell''(\theta)}$$

Consider first the RHS numerator

$$\frac{1}{\sqrt{n}}\ell'(\theta) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \frac{d \ln f_\theta(x_i)}{d\theta} \qquad \text{as indp}$$

We showed earlier that $E\left(\frac{d}{d\theta}\ln f_\theta(x)\right) = 0$ and $\mathrm{Var}\left(\frac{d \ln f_\theta(x)}{d\theta}\right) = I_1(\theta)$ (§2.4), so we have the sum of n iid random variables, which by the CLT converges to a Normal

$$-\frac{1}{\sqrt{n}}\sum_1^n \frac{d \ln f_\theta(x_i)}{d\theta} \longrightarrow N(0, I_1(\theta))$$

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{D} N(0,1)$$

Now let's consider the RHS denumerator

$$\frac{1}{n}\sum_1^n \frac{d^2 \ln f_\theta(x_i)}{d\theta^2} \qquad \frac{1}{n}\sum_1^n X_i \to \mu$$

Again we have the sum of n random variables which are iid with mean $-I_1(\theta)$. Since we are considering their sample mean $\frac{1}{n}\sum_1^n \frac{d^2}{d\theta^2}\ln f_\theta(x_i)$ the strong law of large numbers tells us that

$$-\frac{1}{n}\sum_1^n \frac{d^2}{d\theta^2}\ln f_\theta(x_i) \longrightarrow I_1(\theta)$$

Putting these two results together (Slutsky's thm)

$$\sqrt{n}\,(\hat{\theta}-\theta) \rightarrow \frac{1}{I_1(\theta)}\,N(0, I_1(\theta))$$

or $$(\hat{\theta}-\theta) \rightarrow N\left(0, \frac{1}{n I_1(\theta)}\right)$$

ie $\hat{\theta} \sim N\left(\theta, \frac{1}{n I_1(\theta)}\right)$ asymptotically