A regularized penalty-multiplier method for approximating cavitation solutions with prescribed cavity volume size

Pablo V. Negrón–Marrero · Jey Sivaloganathan

Abstract Let $\Omega \in \mathbb{R}^n$, $n = 2, 3$, be the region occupied by a hyperelastic body in its reference configuration. Let $E(\cdot)$ be the stored energy functional and let $x_0$ be a flaw point in $\Omega$ (i.e., a point of possible discontinuity for admissible deformations of the body). For $V > 0$ fixed, let $u_V$ be a minimizer of $E(\cdot)$ among the set of discontinuous deformations constrained to form a hole of prescribed volume $V$ at $x_0$ and satisfying the homogeneous boundary data $u(x) = Ax$ for $x \in \partial \Omega$. In this paper we describe a regularized penalty–multiplier method for the computation of both $u_V$ and $E(u_V)$ and study its convergence properties. In particular, we show that as the regularization parameter goes to zero, (a subsequence) of the regularized constrained minimizers (computed via a penalty–multiplier method) converge weakly in $W^{1,p}(\Omega \setminus B_\delta(x_0))$ to $u_V$ for any $\delta > 0$. We obtain various sensitivity results for the dependence of the energies and Lagrange multipliers of the regularized constrained minimizers on the boundary data $A$ and on the volume parameter $V$. We show that both the regularized constrained minimizers and $u_V$ satisfy suitable weak versions of the corresponding Euler–Lagrange equations. We also describe the main features of a numerical scheme for approximating $u_V$ and $E(u_V)$ and give numerical examples for the case of a stored energy function of an elastic fluid and in the case of the incompressible limit. The problem considered in this paper is related to, but differs significantly from the standard cavitation problem in which only the homogeneous boundary data is considered.

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P. V. Negrón–Marrero
Department of Mathematics
University of Puerto Rico
Humacao, PR 00791-4300
E-mail: pablo.negron1@upr.edu

J. Sivaloganathan
Department of Mathematical Sciences
University of Bath, Bath
BA2 7AY, UK
E-mail: masjs@bath.ac.uk
boundary data \( A \) is specified. For the standard cavitation problem the authors introduced in an earlier paper [17] the \textit{volume derivative}

\[
G(A) = \lim_{V \to 0^+} \frac{E(u_V) - E(u^h)}{V}, \quad u^h \equiv Ax,
\]

as a tool for characterizing the boundary data leading to discontinuous cavitation solutions. As an application of the results in the current paper we show that the Lagrange multiplier arising from the constrained problem for computing \( u_V \), converges as \( V \searrow 0 \) to the volume derivative.

**Keywords:** nonlinear elasticity, cavitation, regularization, penalty–multiplier method

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1 Introduction

In this paper we consider the problem of numerically computing minimizers within the context of a variational theory of nonlinear elasticity that allows for cavitation. The particular problem we study is that in which the minimizer of the stored energy functional belongs to a set of discontinuous deformations that satisfy homogeneous boundary data and that produce a (not necessarily spherical) hole within the deformed body of a prescribed volume \( V \). The proposed numerical scheme essentially consists of two stages:

1. approximate the original constrained problem by a sequence of \textit{regularized constrained problems} over punctured domains, where the punctures are taken around possible flaw points within the body;
2. each regularized constrained problem is approximated by a sequence of regularized (unconstrained) but penalized problems.

If \( \varepsilon \) represents the diameter of the punctures in the regularized domains and \( \eta \) the penalization parameter, we show the convergence of the numerical approximations to that of the original problem as \( \eta \to \infty \) and \( \varepsilon \searrow 0 \) (in this order).

The problem considered in this paper, though related, differs significantly from that of “standard” cavitation (see [1], [14]) in which just the homogeneous boundary data \( u(\mathbf{x}) = A\mathbf{x} \) for \( \mathbf{x} \in \partial \Omega \), is specified. Depending on the matrix \( A \), the global minimizer may be discontinuous producing a hole or cavitation inside the body. The numerical aspects of cavitation have been studied among others by [9], [10], [15], [7], [11], and [12]. A fundamental problem in studies of cavitation is to analytically or computationally characterize the boundary data \( A \) for which cavitation occurs.

In [17] the authors introduced the concept of the \textit{volume derivative} as a tool for characterizing these boundary displacements.

Central to any scheme for approximating the volume derivative is the computation of constrained minimizers of the type considered in this paper, but for progressively smaller values of the prescribed volume or cavity size \( V \). In [17] the authors proposed
the two stage regularization–penalty method described previously for the computation of the minimizers of these constrained problems, showing the convergence for $\varepsilon$ fixed and with $\eta \to \infty$, and conjecturing the convergence as $\varepsilon \searrow 0$. One purpose of this paper we fill that gap. Moreover, instead of just using a penalty parameter to deal with the volume constraint, we employ a penalty–multiplier technique (also called augmented Lagrangians [13]), that leads to a more stable numerical scheme. An important benefit of the penalty–multiplier technique is that the corresponding Lagrange multipliers converge, as the prescribed cavity size volume tends to zero, to the volume derivative evaluated at the boundary data $A$ (cf. Theorem 6). This approach for computing the volume derivative may provide a more robust scheme compared to the one employed in [17] using difference quotients.

To introduce the results in the paper, consider a nonlinear hyperelastic body occupying the bounded region $\Omega \subset \mathbb{R}^n$ in its reference state. A deformation of the body is a mapping $u : \Omega \to \mathbb{R}^n$ satisfying the local invertibility condition
\[
\det \nabla u(x) > 0 \quad \text{a.e.} \ x \in \Omega.
\]
The energy stored in the deformed body under a deformation $u$ is given by
\[
E(u) = \int_\Omega W(\nabla u(x)) \, dx,
\]where $W : M^{n \times n}_+ \to \mathbb{R}$ is the stored energy function of the material and $M^{n \times n}_+$ denotes the set of $n \times n$ matrices with positive determinant. For a fix matrix $A \in M^{n \times n}_+$, we consider deformations satisfying the displacement boundary condition:
\[
u(x) = Ax \quad \text{for} \ x \in \partial \Omega.
\]
We fix a “flaw” point $x_0 \in \Omega$, and for any fixed $V > 0$ (the prescribed cavity size) we take the admissible set of deformations to be
\[
\mathcal{A}_{A,V} = \{ u \in W^{1,p}(\Omega) \mid \det \nabla u = \det \nabla u \mathcal{L}^n + V \delta_{x_0}, \det \nabla u > 0 \ \text{a.e.}, \ u(x) = Ax \ \text{on} \ \partial \Omega, \ u \text{ satisfies INV on} \ \Omega \},
\]
Here $\det \nabla u$ denotes the distributional determinant of $u$, defined by
\[
< \det \nabla u, \phi > = -\frac{1}{n} \int_\Omega \nabla \phi \cdot (\text{adj} \nabla u)u \, dx, \quad \forall \ \phi \in C_0^\infty(\Omega),
\]
$\mathcal{L}^n$ denotes $n$-dimensional Lebesgue measure, $p > n - 1$, $\delta_{x_0}$ denotes the Dirac measure supported at $x_0 \in \Omega$, and (INV) denotes the condition\(^1\) relating to invertibility introduced in Definition 3.2 of [14]. Results in [20] give conditions on the stored energy function $W$ under which a minimiser for (1) exists on the set $\mathcal{A}_{A,V}$. The results of Henao and Mora-Corral [5] give conditions under which a minimiser also exists in the case $p = n - 1$ and their work in [6] includes justification of the interpretation

\(^1\) For technical reasons, the deformation $u$ has to be extended to a larger domain, whilst still satisfying (INV) on the extended domain, for example by setting it equal to $Ax$ outside $\Omega$ (see [20] for further details). Henceforth we shall assume that all deformations have been extended accordingly without introducing any extra notation.
of $V$ in (3) as the volume of the hole formed by the deformation. Hence, if $u \in \mathcal{A}_V$, then the deformation $u$ produces a hole of volume $V$ in the deformed body.

The requirement that deformations produce a hole of volume $V$ in the deformed body is equivalent to the integral constraint:

$$c(u) \equiv \int_{\Omega} \det \nabla u \, dx - \left( \det A \right) |\Omega| + V = 0.$$  

(5)

(Here $|\Omega|$ is the volume of $\Omega$ and $0 < V < (\det A) |\Omega|$.) Thus we replace the minimization of (1) over (3) with

$$\begin{cases} \min_{u \in \mathcal{A}} E(u), \\ \text{subject to } c(u) = 0, \end{cases}$$

where now

$$\mathcal{A} = \left\{ u \in W^{1,p}(\Omega) \mid \exists \alpha \geq 0 \text{ such that } \text{Det} \nabla u = \det \nabla u \mathbb{R}^n + \alpha \delta_{x_0}, \\ \det \nabla u > 0 \text{ a.e., } u(x) = Ax \text{ on } \partial \Omega, \ u \text{ satisfies INV on } \Omega \right\}.$$  

(7)

(Note that (7) differs from (3) as the volume parameter $V$ is now part of the integral constraint (5).) For any $\varepsilon > 0$, let

$$\Omega_{\varepsilon} = \Omega \setminus B_{\varepsilon}(x_0).$$

(Here and henceforth, we use the notation $B_{\varepsilon}(x_0)$ for the open ball of radius $\varepsilon$ centered at $x_0$.)

The regularized constrained minimization problem is given by:

$$\begin{cases} \min_{u \in \mathcal{A}_{\varepsilon}} E_{\varepsilon}(u), \\ \text{subject to } c_{\varepsilon}(u) = 0, \end{cases}$$

where

$$E_{\varepsilon}(u) = \int_{\Omega_{\varepsilon}} W(\nabla u(x)) \, dx, \quad c_{\varepsilon}(u) = \int_{\Omega_{\varepsilon}} \det \nabla u \, dx - \left( \det A \right) |\Omega| + V,$$

and

$$\mathcal{A} = \left\{ u \in W^{1,p}(\Omega_{\varepsilon}) \mid \det \nabla u = (\det \nabla u) \mathbb{R}^n, \det \nabla u > 0 \text{ a.e.,} \\ u(x) = Ax \text{ on } \partial \Omega, \ u \text{ satisfies INV} \right\},$$

We set $\mathcal{A}_0 = \mathcal{A}$ and $c_0 = c$, and define the sets

$$\mathcal{C}_\varepsilon = \{ u \in \mathcal{A} \mid c_{\varepsilon}(u) = 0 \}.$$

**Remark 1** The hypotheses and results of [20] are easily adapted to prove that a (not necessarily unique) minimiser $u_{V,\varepsilon}$ of $E_{\varepsilon}$ on $\mathcal{A}_{\varepsilon}$ exists for each $\varepsilon \geq 0$ and $V > 0$ small enough.

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2 In Appendix A we show that the admissible sets $\mathcal{C}_\varepsilon$ are nonempty for $\varepsilon$ sufficiently small.
To compute approximations of the constrained problem (8), we use a penalty–multiplier method in which the energy functional in (8) is replaced by:

$$E_{\varepsilon, \mu, \eta}(u) = E_\varepsilon(u) + \mu c_\varepsilon(u) + \frac{1}{2} \eta c_\varepsilon(u)^2.$$  

(9)

Here \(\eta\) is a “large” positive parameter and \(\mu \in \mathbb{R}\). Thus we replace the regularized constrained problem (8) with the regularized “unconstrained” problem:

$$\inf_{u \in H_0^1(\Omega)} E_{\varepsilon, \mu, \eta}(u).$$  

(10)

In Proposition 1 we show existence of a minimizer \(u_{\varepsilon, \mu, \eta}\) for (10). This result is then used in Theorem 1 to show that for fixed \(\varepsilon, V > 0\), there exist sequences \(\{\mu_j\}\), \(\{\eta_j\}\) such that \(\{u_{\varepsilon, \mu_j, \eta_j}\}\) converges weakly in \(W^{1,p}(\Omega)\) to a solution \(u_{\varepsilon, V}\) of (8), and with \(c_\varepsilon(u_{\varepsilon, \mu_j, \eta_j}) \to 0\). In Theorem 2 we establish a result on the weak form of the Euler–Lagrange equations for the minimizer \(u_{\varepsilon, V}\). This result is then used to study the sensitivity of the minimum energy \(E_\varepsilon(u_{\varepsilon, V})\) and its corresponding Lagrange multiplier, with respect to variations in the boundary data \(A\) and the volume \(V\) (Theorem 3).

In Section 3 we prove several key results that will be used as the basis for a numerical scheme for computing a minimizer \(u_V\) of (6). First in Theorem 4 we show that for a sequence \(\{\varepsilon_j\}\) converging to zero, a subsequence of the corresponding regularized constrained minimizers \(\{u_{\varepsilon_j, \mu_j, \eta_j}\}\) converges weakly in \(W^{1,p}(\Omega_\delta)\) to a solution \(u_V\) of (6), for any \(\delta > 0\). The main difficulty in this proof is to show that the limiting function \(u_V\) is a solution of (6), in particular that it satisfies the integral volume constraint in (6). A key ingredient to obtain this result is Lemma 1 which shows that any function \(u \in C^0_A\) can be approximated, together with its energy, by functions in \(C_\varepsilon A\). Two other important results in Section 3 are, firstly, on the weak form of the Euler–Lagrange equations satisfied by the minimizer \(u_V\) (Theorem 5) and, secondly, a result on the convergence to the volume derivative as \(V \searrow 0\) of the Lagrange multiplier \(\mu_V\) corresponding to the volume constraint on \(u_V\) (see Theorem 6). A major technical difficulty in obtaining some of the results in this section is due to the fact that the domains of the sequence of approximating functions are changing around the possible flaw point \(x_0\) with the sequential index, thus complicating the verification of various estimates and certain limits of weakly converging sequences.

A simple class of polyconvex isotropic stored energy functions to which the results in this paper can be applied is given by

$$W(F) = \frac{\kappa}{q} ||F||^q + h(\text{det}F),$$  

(11)

where \(\kappa > 0\), \(q \in [n-1, n]\) and \(h : (0, \infty) \to (0, \infty)\) is such that

1. \(h\) is a \(C^2\), convex function and
2. \(h(\delta) \to \infty\) and \(\frac{h(\delta)}{\delta} \to \infty\) as \(\delta \to 0, \infty\) respectively.

(12a)

(12b)

However, we note that the results of this paper are readily extended to apply to more general polyconvex stored energy functions under varied hypotheses.
In Section 4 we describe the main features of a numerical scheme for approximating solutions of the problem (6). Then we use this scheme in a numerical example for the case of an elastic fluid (corresponding to \( \kappa = 0 \) in (11)). For this class of materials and for a spherical domain, an exact solution of (6) is known and we can thus check the various convergence results in the paper in this case. We also report some simulations for the so called incompressible limit case. Here, we add a term of the form \( k(\det F - 1)^2 \) to (11) (with \( \kappa > 0 \)), where \( k > 0 \) is a given constant and, although the solutions of the intermediate problems with \( k \) given are not known explicitly, the limiting case \( (k \to \infty) \) corresponds to an incompressible material for which the solution is known explicitly. Thus, in this case, we can test the robustness of the penalty–multiplier scheme by computing the solutions of several intermediate volume constrained problems (with \( V \) fixed but \( k \) varying), and test for convergence of the computed solutions to the limiting incompressible solution as \( k \) gets large.

2 The regularized penalty–multiplier method

In this section we study the unconstrained problem (10), in particular the convergence properties of the penalty multiplier method. We assume that the stored energy function \( W(F) \) satisfies the following:

H1: (Polyconvexity) There exists with \( G : (M_n^{n \times n})^{n-1} \times (0, \infty) \to \mathbb{R} \) continuous and convex such that

\[
W(F) = \begin{cases} 
G(F, \det F), & n = 2, \\
G(F, \text{adj} F, \det F), & n = 3.
\end{cases}
\]

H2: (Growth) There exists \( p \in (n-1, n), c_1 > 0 \), and a \( C^2 \) function \( h \) such that

\[
W(F) \geq c_1 |F|^p + h(\det F) \quad \text{for} \quad F \in M_n^{n \times n},
\]

where the function \( h \) satisfies conditions (12).

We begin by showing the existence of minimizers for problem (10). However, we first note that because of the boundary and INV conditions for functions \( u \in A_{\varepsilon} \), we have

\[
-|\det A| |\Omega| + V \leq c_\varepsilon(u) \leq \int_{\Omega \varepsilon} \det(\nabla u(x)) dx \leq (|\det A| |\Omega|),
\]

by the non–negativity of the determinant and so \( c_\varepsilon(u) \) is (uniformly) bounded on \( A_{\varepsilon} \).

**Proposition 1** For any \( \mu \in \mathbb{R} \), \( \eta > 0 \), there exists a minimizer \( u_{\varepsilon, \mu, \eta} \in A_{\varepsilon} \) of \( E_{\varepsilon, \mu, \eta}(u) \) on \( A_{\varepsilon} \). Moreover, for any \( \delta > 0 \), the parameter \( \eta \) can be chosen sufficiently large such that the minimizer \( u_{\varepsilon, \mu, \eta} \) satisfies that \( |c_\varepsilon(u_{\varepsilon, \mu, \eta})| < \delta \).

**Proof** Since the homogeneous deformation \( u = Ax \) lies in \( A_{\varepsilon} \),

\[
g^* := \inf_{u \in A_{\varepsilon}} E_{\varepsilon, \mu, \eta}(u) < \infty.
\]

By the non–negativity of \( W \) and since \( \eta > 0 \), we obtain \( E_{\varepsilon, \mu, \eta}(u) \geq \mu c_\varepsilon(u) \) for all \( u \in A_{\varepsilon} \). By the uniform boundedness of \( c_\varepsilon(u) \) mentioned above, it follows that \( g^* \neq \infty. \)
Let now \( \{u_k\} \) in \( \mathcal{A}_\Omega^\varepsilon \) be a minimizing sequence, i.e., \( E_{\varepsilon, \mu, \eta}(u_k) \to g^* \). Since \( \{\mu c_\varepsilon(u_k)\} \) is bounded, we can find an \( L > 0 \) such that \( \mu c_\varepsilon(u_k) \geq -L \) for all \( k \). Thus, for \( k \) sufficiently large, we obtain
\[
\int_{\Omega_\varepsilon} W(\nabla u_k(x)) \, dx - L \leq g^* + 1.
\]
It follows now from the growth hypotheses (H1)–(H2) that there exists a subsequence \( \left\{ u_{k_j} \right\} \) which converges weakly in \( W^{1,p}(\Omega_\varepsilon) \) to a function \( u^* \), and that \( \left\{ \det \nabla u_{k_j} \right\} \) converges weakly in \( L^1(\Omega_\varepsilon) \) to a function \( \theta \). Since \( p \in (n - 1, n) \), it follows from [14, Theorem 4.2], that \( u^* \) satisfies condition INV, \( \theta = \det \nabla u^* \), and \( \det \nabla u^* > 0 \) almost everywhere. Thus \( u^* \in \mathcal{A}_\Omega^\varepsilon \).

Upon adapting the lower continuity results in [20], it follows that \( E_{\varepsilon, \mu, \eta} \) is sequentially weakly lower semicontinuous. Thus we have that
\[
E_{\varepsilon, \mu, \eta}(u^*) \leq \liminf_{j \to \infty} E_{\varepsilon, \mu, \eta}(u_{k_j}) = g^*,
\]
i.e., that \( u_{W, \varepsilon, \mu, \eta} \equiv u^* \in \mathcal{A}_\Omega^\varepsilon \) is a minimizer.

For the last part of the proposition, we argue by contradiction. Suppose that for some \( \delta_0 \) there exists a sequence \( \eta_j \to \infty \) such that the corresponding minimizers \( \{u_j\} \) satisfy \( c_\varepsilon(u_j) \geq \delta_0 \) for all \( j \). Note that for all \( j \),
\[
E_{\varepsilon, \mu, \eta_j}(u_j) \leq f^*_{\varepsilon},
\]
where \( f^*_{\varepsilon} \) is the minimum value in (8) (cf. Remark 1). Since \( \{\mu c_\varepsilon(u_j)\} \) is bounded, we can find \( L > 0 \) such that \( \mu c_\varepsilon(u_j) \geq -L \) for all \( j \). Hence
\[
f^*_{\varepsilon} \geq \mu c_\varepsilon(u_j) + \frac{1}{2} \eta_j c_\varepsilon(u_j)^2 \geq -L + \frac{1}{2} \eta_j \delta_0^2 \to \infty,
\]
which leads to a contradiction, completing the proof.

We now show how to construct sequences \( \{\mu_j\} \) and \( \{\eta_j\} \) and give hypotheses under which, the computed minimizers in (10), converge to a solution of (8).

**Theorem 1** Let the stored energy function \( W \) satisfy the conditions H1–H2. Let \( \gamma \in (0, 1), \beta > 1, \eta_1 > 0, \mu_1 \in \mathbb{R}, \) and \( u_0 \in \mathcal{A}_\Omega^\varepsilon \) be given. Let the sequences \( \{\mu_j\}, \{\eta_j\}, \) and \( \{u_j\} \) be given by:
\[
E_{\varepsilon, \mu_j, \eta_j}(u_j) = \min_{u \in \mathcal{A}_\Omega^\varepsilon} E_{\varepsilon, \mu_j, \eta_j}(u),
\]
\[
\mu_{j+1} = \mu_j + \eta_j c_\varepsilon(u_j),
\]
\[
\eta_{j+1} = \begin{cases} \eta_j, & \text{if } |c_\varepsilon(u_j)| \leq \gamma |c_\varepsilon(u_{j-1})|, \\ \beta \eta_j, & \text{otherwise}. \end{cases}
\]
Assume that \( \{\mu_j\} \) is bounded. Then \( c_\varepsilon(u_j) \to 0 \), and \( \{u_j\} \) has a subsequence \( \{u_{j_k}\} \) that converges weakly in \( W^{1,p}(\Omega_\varepsilon) \) to a minimizer \( u_\varepsilon \) of problem (8) and with
\[
E_{\varepsilon}(u_\varepsilon) = \liminf_k E_{\varepsilon, \mu_{j_k}, \eta_{j_k}}(u_{j_k}).
\]
Proof By Proposition 1, a function \( u_j \in \mathcal{A}_\lambda^E \) satisfying (15a) exists for each \( j \). From (14) we get that
\[
E_{\varepsilon, \mu_j, \eta_j}(u_j) \leq f_\varepsilon^*, \quad \forall j.
\]
From this inequality and using that \( W \) is nonnegative, we get that
\[
\mu_j c_\varepsilon(u_j) + \frac{1}{2} \eta_j c_\varepsilon(u_j)^2 \leq f_\varepsilon^*, \quad \forall j.
\]
(17)

Note that the sequence \( \{\eta_j\} \) is increasing. Thus in (15c) we have two possibilities:

1. the sequence \( \{\eta_j\} \) remains bounded, in which case, \( |c_\varepsilon(u_j)| \leq \gamma |c_\varepsilon(u_{j-1})| \) is satisfied for all but finitely many indexes \( j \). Clearly \( c_\varepsilon(u_j) \to 0 \) in this case.

2. Otherwise (for a subsequence) \( \eta_j \to \infty \), in which case (17) and the boundedness of \( \{\mu_j\} \) would imply that \( c_\varepsilon(u_j) \to 0 \).

Thus, in both cases, we have that \( c_\varepsilon(u_j) \to 0 \).

If \( \mu_j c_\varepsilon(u_j) \geq -L \) for all \( j \), where \( L > 0 \), then from (14) we get that
\[
\int_{\Omega_\varepsilon} W(\nabla u_j(x)) \, dx - L \leq f_\varepsilon^*.
\]
By the arguments in the proof of Proposition 1, there exists a subsequence \( \{u_{j_k}\} \) which converges weakly in \( W^{1,p}(\Omega_\varepsilon) \) to a function \( u_\varepsilon \), and such that \( \det \nabla u_{j_k} \) converges weakly in \( L^1(\Omega_\varepsilon) \) to \( \det \nabla u_\varepsilon \), where \( u_\varepsilon \) satisfies condition INV and \( \det \nabla u_\varepsilon > 0 \) almost everywhere. Thus \( u_\varepsilon \in \mathcal{A}_\lambda^E \) and \( c_\varepsilon(u_\varepsilon) = 0 \). Moreover, since \( \mu_j c_\varepsilon(u_j) \to 0 \) by the assumed boundedness of \( \{\mu_j\} \), we have that
\[
f_\varepsilon^* \leq E_\varepsilon(u_\varepsilon) \leq \liminf_k E_{\varepsilon, \mu_{j_k}, \eta_{j_k}}(u_{j_k}) \leq f_\varepsilon^*.
\]
It follows that \( u_\varepsilon \) is a minimizer of (8) and that (16) holds. \( \square \)

Remark 2 The multiplier iteration (15b) is the most common type of iteration used in the augmented Lagrangian scheme. The motivation for this formula comes from the observation that the multiplier for the problem \( \inf_{u \in \mathcal{A}_\lambda^E} E_{\varepsilon, \mu_j} \Phi(u) \) is \( \mu_\varepsilon - \mu_j \) where \( \mu_\varepsilon \) is the Lagrange multiplier corresponding to the problem (8). On the other hand, since \( E_{\varepsilon, \mu_j, \eta_j}(\cdot) \) is a quadratic penalty function for this problem, one expects \( \eta_j c_\varepsilon(u_j) \) to be close to \( \mu_\varepsilon - \mu_j \) for \( \eta_j \) sufficiently large. Hence \( \mu_\varepsilon - \mu_j \approx \eta_j c_\varepsilon(u_j) \) from which (15b) follows. (See [8], [13, Pages 451–452].)

Remark 3 The assumption of boundedness on the multiplier sequence \( \{\mu_j\} \) is typical of local convergence results for the augmented Lagrangian scheme (cf. [2]). One could in practice enforce this condition by requiring that the iterates remain on a prescribed bounded interval. However, if this interval does not contain the actual multiplier \( \mu_\varepsilon \) of the problem (8), then this would impede the convergence of \( \{\mu_j\} \) to \( \mu_\varepsilon \). A better practice is just to monitor the growth of the \( \mu_j \) to detect some possible tendency to unboundedness.

Our next results give conditions under which the minimizer \( u_\varepsilon \) in (8), satisfies a weak form of the Euler-Lagrange equations for this problem. We use the following modified version of hypothesis H2 for the term \( \tilde{W} \) in the stored energy function in the next theorem:
H3: (Growth) There exists a $C^2$ function $h$ such that
\[ W(F) \geq h(\det F) \quad \text{for} \quad F \in M_{+}^{n \times n}, \]
where the function $h$ satisfies conditions (12).

**Theorem 2** Let $\{u_j\}$ be the sequence of Theorem 1 generated according to (15), and $\{u_{j_k}\}$ a subsequence that converges weakly in $W^{1,p}(\Omega_\varepsilon)$ to a solution $u_\varepsilon$ of (8). Assume that the stored energy function $W$ is uniformly quasiconvex of the form $\gamma |F|^p + W(F)$ where $\gamma > 0$ and $W$ satisfy H1 and H3. Furthermore, assume there exist constants $K, \varepsilon_0 > 0$ such that:
\[ \left| \frac{dW}{dF}(CF)F^T \right| \leq K[W(F) + 1] \quad \text{for all} \quad F \in M_{+}^{n \times n}, \tag{18} \]
whenever $|C - I| < \varepsilon_0$. Then $\{\mu_j\}$ has a subsequence converging to $\mu_\varepsilon$, where
\[ \int_{\Omega_\varepsilon} \left[ \frac{dW}{dF}(\nabla u_\varepsilon) + \mu_\varepsilon (\text{adj} \nabla u_\varepsilon)^T \right] \cdot \nabla [v(u_\varepsilon)] \, dx = 0, \tag{19} \]
for all $v \in C^1(\mathbb{R}^n)$ with $v = 0$ on $\mathbb{R}^n \setminus \mathcal{E}$, where $\mathcal{E} = \{Ax : x \in \Omega\}$. Moreover if $u_\varepsilon \in C^2(\Omega_\varepsilon) \cap C^1(\overline{\Omega_\varepsilon})$ with $\det \nabla u_\varepsilon > 0$ in $\Omega_\varepsilon$, then
\[ \text{div} \left[ \frac{dW}{dF}(\nabla u_\varepsilon) + \mu_\varepsilon (\text{adj} \nabla u_\varepsilon)^T \right] = 0, \quad \text{in} \quad \Omega_\varepsilon, \tag{20a} \]
\[ u_\varepsilon(x) = Ax \quad \text{on} \quad \partial \Omega, \tag{20b} \]
\[ \left[ \frac{dW}{dF}(\nabla u_\varepsilon) + \mu_\varepsilon (\text{adj} \nabla u_\varepsilon)^T \right] n = 0 \quad \text{on} \quad \partial B(x_0), \tag{20c} \]
\[ \int_{\Omega_\varepsilon} \det \nabla u_\varepsilon \, dx = (\det A) |\Omega| - V. \tag{20d} \]

**Proof** To show (19), we first derive the corresponding equilibrium equation for each $u_j$. We use variations of $u_j$ of the form $u = u_j + sv(u_j)$ where $v \in C^1(\mathbb{R}^n)$ with $v = 0$ on $\mathbb{R}^n \setminus \mathcal{E}$. From [20, Corollary 6.4] it follows that for $s$ sufficiently small, the function $u_\varepsilon \in W^{1,n}$. (Note that the variation $u_\varepsilon$ is not required to satisfy the constraint $c_\varepsilon(u) = 0$ as $u_j$ is a solution of an unconstrained problem.) To show (19) for $u_j$, first note that
\[ \int_{\Omega_\varepsilon} [W(\nabla u_j) - W(\nabla u_j)] \, dx = s \int_{\Omega_\varepsilon} \left[ \int_0^1 \frac{dW}{dF}(I + st \nabla v(u_j)) |\nabla u_j| \, dr \right] \cdot \nabla v(u_j) \, dx \]
It follows now from (18) that for $s$ small enough,
\[ \left| \int_0^1 \frac{dW}{dF}(I + st \nabla v(u_j)) |\nabla u_j| \, dr \right| \leq K[W(\nabla u_j) + 1] \in L^1(\Omega_\varepsilon). \]

$^{3}$ $\mu_\varepsilon$ is the Lagrange multiplier corresponding to the volume constraint in (8) and is a measure of the Cauchy stress acting on the deformed inner cavity (cf. (49)).
Upon invoking the Dominated Convergence Theorem, we obtain
\[
\lim_{s \to 0} \frac{1}{s} \int_{\Omega_{\varepsilon}} [W(\nabla u_s) - W(\nabla u_j)] \, dx = \int_{\Omega_{\varepsilon}} \frac{dW}{dF}(\nabla u_j) \nabla u_j^T \cdot \nabla v(\nabla u_j) \, dx \tag{21}
\]
Also
\[
\mu_j[c_{\varepsilon}(u_j) - c_{\varepsilon}(u)] + \frac{1}{2} \eta_j[c_{\varepsilon}^2(u_j) - c_{\varepsilon}^2(u)]
= [\mu_j + \frac{1}{2} \eta_j(c_{\varepsilon}(u_j) + c_{\varepsilon}(u))] [c_{\varepsilon}(u_j) - c_{\varepsilon}(u)].
\]
Now
\[
c_{\varepsilon}(u_j) - c_{\varepsilon}(u_j) = s \int_{\Omega_{\varepsilon}} \left[ \int_{0}^{1} \left[ \text{adj}(I + st\nabla v(u_j)) \right] \cdot \nabla v(\nabla u_j) \right] \, dx.
\]
It follows now since \( v \in C^1(\mathbb{R}^n) \) with \( v = 0 \) on \( \mathbb{R}^n \setminus \mathcal{E} \), that
\[
\lim_{s \to 0} \frac{1}{s} [c_{\varepsilon}(u_j) - c_{\varepsilon}(u_j)] = \int_{\Omega_{\varepsilon}} [I \cdot \nabla v(\nabla u_j)] \, dx.
\]
Combining this with (21) and using that \( c_{\varepsilon}(u_j) \to c_{\varepsilon}(u_j) \) as \( s \to 0 \), we get that
\[
\frac{d}{ds} E_{\varepsilon, \mu_j, \eta_j}(u_j) \bigg|_{s=0} = \int_{\Omega_{\varepsilon}} \frac{dW}{dF}(\nabla u_j) \nabla u_j^T \cdot \nabla v(\nabla u_j) \, dx.
\]
Since \( u_j \) is a minimizer, we must have that
\[
\int_{\Omega_{\varepsilon}} \left[ \frac{dW}{dF}(\nabla u_j) \nabla u_j^T + (\mu_j + \eta_j c_{\varepsilon}(u_j))(\det \nabla u_j) \right] \cdot \nabla v(\nabla u_j) \, dx = 0, \tag{22}
\]
for all such \( v \)'s. We now drop to the subsequence \( \{u_{j_k}\} \) that converges weakly in \( W^{1,p}(\Omega_{\varepsilon}) \) to \( u_{\varepsilon} \) and with \( \det \nabla u_{j_k} \to \det \nabla u_{\varepsilon} \) in \( L^1(\Omega_{\varepsilon}) \). Furthermore, because of (16), we may assume that the subsequence is such that
\[
E_{\varepsilon}(u_{\varepsilon}) = \lim_{k} E_{\varepsilon, \mu_{j_k}, \eta_{j_k}}(u_{j_k}),
\]
and, by the boundedness of \( \{\mu_j\} \), that \( \mu_{j_k} \to \mu_{\varepsilon} \). Thus
\[
\lim_{k} \left[ \mu_{j_k} c_{\varepsilon}(u_{j_k}) + \frac{1}{2} \eta_{j_k} c_{\varepsilon}(u_{j_k})^2 \right] = 0.
\]
It follows now that
\[
E_{\varepsilon}(u_{\varepsilon}) - \gamma \int_{\Omega_{\varepsilon}} |\nabla u_{\varepsilon}|^p \, dx = \int_{\Omega_{\varepsilon}} \tilde{W}(\nabla u_{\varepsilon}) \, dx,
\]
\[
\leq \liminf_{k} \left[ \int_{\Omega_{\varepsilon}} \tilde{W}(\nabla u_{j_k}) \, dx + \mu_{j_k} c_{\varepsilon}(u_{j_k}) + \frac{1}{2} \eta_{j_k} c_{\varepsilon}(u_{j_k})^2 \right],
\]
\[
= \lim_{k} E_{\varepsilon, \mu_{j_k}, \eta_{j_k}}(u_{j_k}) - \gamma \limsup_{k} \int_{\Omega_{\varepsilon}} |\nabla u_{j_k}|^p \, dx,
\]
from which we obtain
\[ \limsup_k \int_{\Omega} |\nabla u_j|^p \, dx \leq \int_{\Omega} |\nabla u_\varepsilon|^p \, dx. \]

This together with the weak convergence of \( \{ u_j \} \) to \( u_\varepsilon \) in \( W^{1,p}(\Omega_\varepsilon) \), implies the strong convergence \( u_j \to u_\varepsilon \) in \( W^{1,p}(\Omega_\varepsilon) \). Thus, we may assume that \( \{ u_j \} \) and \( \{ \nabla u_j \} \) converge almost everywhere to \( u_\varepsilon \) and \( \nabla u_\varepsilon \) respectively. Thus using (18) and the Dominated Convergence Theorem in (22) (dropping to the subsequence \( \{ u_{j_k} \} \)), we obtain
\[ \int_{\Omega_\varepsilon} \left[ \frac{dW}{dF}(\nabla u_\varepsilon) \nabla u_\varepsilon^T + \mu_\varepsilon (\det \nabla u_\varepsilon) I \right] \cdot \nabla v(u_\varepsilon) \, dx = 0. \]

Since \( (\det \nabla u_\varepsilon) I = (\adj \nabla u_\varepsilon)^T \nabla u_\varepsilon^T \) and \( \nabla [v(u_\varepsilon)] = \nabla v(u_\varepsilon) \nabla u_\varepsilon \), it follows that the above equation is equivalent to (19).

Now assume that \( u_\varepsilon \in C^2(\Omega_\varepsilon) \cap C^1(\overline{\Omega_\varepsilon}) \) with \( \det \nabla u_\varepsilon > 0 \) in \( \Omega_\varepsilon \). Note that (20b) and (20d) follow from the fact that \( u_\varepsilon \) is a solution of (8). The proof that (20a) holds is similar to the one given in [20, Theorem 5.1] and thus we omit it. Now multiply (20a) by \( v(u_\varepsilon) \) where \( v \in C^1(\mathbb{R}^n) \) with \( v = 0 \) on \( \mathbb{R}^n \setminus \delta \), and integrate by parts using (19) to get that
\[ \int_{\partial \Omega_\varepsilon} v(u_\varepsilon) \cdot \left[ \frac{dW}{dF}(\nabla u_\varepsilon) + \mu_\varepsilon (\adj \nabla u_\varepsilon)^T \right] n \, ds(y) = 0. \]

Since the normal \( n \) to \( \partial \Omega_\varepsilon \) is mapped by \( u_\varepsilon \) to
\[ \tilde{n}(u_\varepsilon) = (\det \nabla u_\varepsilon) (\nabla u_\varepsilon)^{-T} n, \]

upon setting \( y = u_\varepsilon(x) \), the previous equation is equivalent to:
\[ \int_{u_\varepsilon(\partial \Omega_\varepsilon)} v(y) \cdot [T(y) + \mu_\varepsilon I] \tilde{n}(y) \, ds(y) = 0, \tag{23} \]
where the Cauchy stress tensor \( T(u_\varepsilon) \) is given by
\[ T(u_\varepsilon) = (\det \nabla u_\varepsilon)^{-1} \frac{dW}{dF}(\nabla u_\varepsilon)(\nabla u_\varepsilon)^T. \]

From (23) and the arbitrariness of \( v \), we get that
\[ [T(y) + \mu_\varepsilon I] \tilde{n}(y) = 0, \quad \forall \ y \in u_\varepsilon(\partial \Omega_\varepsilon(x_0)). \]

which after changing variables back to \( \Omega_\varepsilon \) yields (20c).

Remark 4 The hypotheses on \( W \) in Theorem 2 are satisfied by the model stored energy function (11). The argument used in the proof of this theorem to get the strong convergence of the subsequence \( \{ u_{j_k} \} \) to \( u_\varepsilon \) in \( W^{1,p}(\Omega_\varepsilon) \) from its weak convergence, is a slight variation of the one due to Evans [3].
We now study the sensitivity of the attained minimum value in (8) with respect to changes in the matrix $A$ and the volume parameter $V$. In the usual sensitivity theorems of optimization theory, the parameters that change are on the right hand sides of the constraints which is the case for $V$ in our problem. As for the matrix $A$, it appears both in the right hand side of the volume constraint and in the displacement boundary condition on $\partial \Omega$. Thus our calculation picks up an additional term from $\partial \Omega$. We use the notation $u_e(\cdot, A, V)$ to emphasize the dependence of the minimizer on both $A$ and $V$.

**Theorem 3** Let $u_e(\cdot, A, V)$ be a minimizer in (6) and assume that $u_e(\cdot, A, V) \in C^2(\Omega_e) \cap C^1(\overline{\Omega}_e)$ and that $u_e \in C^2(\Omega_e \times M_{n \times n}^+ \times (0, V_0))$ for some $V_0 > 0$. Then for $A = \text{diag}(\lambda_1, \ldots, \lambda_n)$, $\lambda_i > 0$ for all $i$, we have that

$$
\frac{\partial}{\partial \lambda_i} E_e(u_e(\cdot, A, V)) = \int_{\partial \Omega} x_i e_i \cdot \left[ \frac{dW}{dF} (\nabla u_e) + \mu_e (\text{adj} \nabla u_e)^T \right] n ds - \mu_e |\Omega| \frac{\det A}{\lambda_i}, \quad i = 1, \ldots, n,
$$

$$
\frac{\partial}{\partial V} E_e(u_e(\cdot, A, V)) = \mu_e.
$$

where $\{e_i\}$ is the standard basis of $\mathbb{R}^n$. Moreover with the “dots” denoting derivatives with respect to $V$, we have:

$$
\frac{\partial \mu_e}{\partial V} = \int_{\Omega_e} \nabla \bar{u}_e \cdot C(\nabla u_e) \nabla \bar{u}_e dx - \mu_e \int_{\Omega_e} (\text{adj} \nabla u_e)^T \nabla \bar{u}_e dx,
$$

$$
\frac{\partial \mu_e}{\partial \lambda_i} = \int_{\partial \Omega} x_i e_i \cdot \left[ C(\nabla u_e) |\nabla u_e| + \frac{\partial \mu_e}{\partial V} (\text{adj} \nabla u_e)^T \right] n ds + \mu_e \int_{\partial \Omega} x_i e_i \cdot \left[ (\text{adj} \nabla u_e)^T \nabla \bar{u}_e \right] - (\text{adj} \nabla u_e)^T \nabla \bar{u}_e^T \nabla u_e \right] - |\Omega| \frac{\det A}{\lambda_i} \frac{\partial \mu_e}{\partial V},
$$

where $C(F)$ denotes the elasticity tensor (fourth order) at $F$.

**Proof** Let $u_{e,i} = \frac{\partial u_e}{\partial \lambda_i}$. By the assumed smoothness on $u_e$, we have that

$$
\frac{\partial}{\partial \lambda_i} E_e(u_e(\cdot, A, V)) = \int_{\partial \Omega_e} W(\nabla u_e) dx = \int_{\Omega_e} \frac{dW}{dF} (\nabla u_e) \cdot \nabla u_{e,j} dx.
$$

If we multiply (20a) by $u_{e,j}$, integrate by parts, and use the boundary condition (20c), we get that

$$
\int_{\Omega_e} \left[ \frac{dW}{dF} (\nabla u_e) + \mu_e (\text{adj} \nabla u_e)^T \right] \cdot \nabla u_{e,j} dx = \int_{\partial \Omega} u_{e,i} \cdot \left[ \frac{dW}{dF} (\nabla u_e) + \mu_e (\text{adj} \nabla u_e)^T \right] n ds.
$$
Since (20b) implies that $u_{e,i}(x) = x_i e_i$ (no summation) for $x \in \partial \Omega$, the above equation can be written as

$$\int_{\Omega_e} \frac{dW}{dF} (\nabla u_e) \cdot \nabla u_{e,i} \, dx = \int_{\partial \Omega} x_i e_i \cdot \left[ \frac{dW}{dF} (\nabla u_e) + \mu_e (\text{adj} \nabla u_e)^T \right] n \, ds$$

$$- \mu_e \int_{\Omega_e} (\text{adj} \nabla u_e)^T \cdot \nabla u_{e,i} \, dx. \quad (27)$$

We now differentiate (20d) with respect to $\lambda_i$ to get that

$$\int_{\Omega_e} (\text{adj} \nabla u_e)^T \cdot \nabla u_{e,i} \, dx = |\Omega| \frac{\partial}{\partial \lambda_i} (\det A) = |\Omega| \frac{1}{\lambda_i} \det A.$$

Combining this with (26) and (27), gives the result (24a).

For (24b) we let $u_e = \frac{\partial w_e}{\partial V}$. We now have that

$$\frac{\partial}{\partial V} E_e(u_e(\cdot, A, V)) = \int_{\Omega_e} \frac{dW}{dF} (\nabla u_e) \cdot \nabla u_e \, dx.$$

Since (20b) implies that $u_e(x) = 0$ on $\partial \Omega$, the equivalent of (27) is now

$$\int_{\Omega_e} \frac{dW}{dF} (\nabla u_e) \cdot \nabla u_e \, dx = -\mu_e \int_{\Omega_e} (\text{adj} \nabla u_e)^T \cdot \nabla u_e \, dx. \quad (28)$$

Differentiating (20d) with respect to $V$, we get that

$$\int_{\Omega_e} (\text{adj} \nabla u_e)^T \cdot \nabla u_e \, dx = -1,$$

which combined with the previous equation yields that

$$\mu_e = \int_{\Omega_e} \frac{dW}{dF} (\nabla u_e) \cdot \nabla u_e \, dx = \frac{\partial}{\partial V} E_e(u_e(\cdot, A)). \quad (29)$$

Now for the second part of the theorem, we differentiate with respect to $V$ both sides of the first equation in (29), to get that

$$\frac{\partial \mu_e}{\partial V} = \int_{\Omega_e} \nabla u_e \cdot C(\nabla u_e) [\nabla \tilde{u}_e] \, dx + \int_{\Omega_e} \frac{dW}{dF} (\nabla u_e) \cdot \nabla u_e \, dx.$$

An argument similar to the one leading to (28), using now that that $\tilde{u}_e = 0$ on $\partial \Omega$, yields that

$$\int_{\Omega_e} \frac{dW}{dF} (\nabla u_e) \cdot \nabla u_e \, dx = -\mu_e \int_{\Omega_e} (\text{adj} \nabla u_e)^T \cdot \nabla u_e \, dx.$$

Combining this with the previous equation we get (25a). Now for the partial derivatives of $\mu_e$ with respect to the $\lambda_i$, we use (24b) to get that:

$$\frac{\partial \mu_e}{\partial \lambda_i} = \frac{\partial}{\partial \lambda_i} \left[ \frac{\partial}{\partial V} E_e(u_e(\cdot, A, V)) \right].$$

Equation (25b) now follows from (24a) and the identity:

$$\frac{d}{dF} (\text{adj} F)^T [H] = \left[ ((\text{adj} F)^T \cdot H) I - (\text{adj} F)^T H F \right] F^{-T}.$$
3 Convergence of the regularized constrained minimizers

We now show that the regularized constrained minimizers whose existence is given by Theorem 1, converge to a solution of the “non–regular” constrained problem (6). The first part of the proof of this result, dealing with the convergence and the existence of the limit, is very similar to that in [21, Theorem 4.1] and consequently we sketch most of it. The second part in which we show that the limiting function is actually a solution of (6) is more subtle, again due to the treatment of the integral volume constraint in (6). In particular we need the following result which shows that given any function $u \in \mathcal{C}_A^0$ and any sequence $\varepsilon_j \to 0$, we can approximate the energy of $u$ with the corresponding energies of a sequence of functions $\{\hat{u}_j\}$ where $\hat{u}_j \in \mathcal{C}_{\varepsilon_j}^A$ for all $j$.

**Lemma 1** Let $\Omega$ be a bounded, open set, and let the stored energy function $W$ satisfy the conditions H1–H2. Let $u \in \mathcal{C}_A^0$ and $V \in (0, |\Omega| \det A)$. Then for any for any sequence $\{\varepsilon_j\}$ with $\varepsilon_j \to 0$, there exists a sequence of functions $\{\hat{u}_j\}$ in $W^{1,p}(\Omega)$ with $\hat{u}_j|_{\Omega_j} \in \mathcal{C}_{\varepsilon_j}^A$ for each $j$, and such that

$$\lim_{j \to \infty} \int\Omega W(\nabla \hat{u}_j(x)) \, dx = \int\Omega W(\nabla u(x)) \, dx.$$ 

**Proof** For any $0 < \eta < 1$ and $\varepsilon > 0$, we let

$$\Omega^\eta = \{x : \eta^{-1}(x - x_0) + x_0 \in \Omega\} = \eta(\Omega - x_0) + x_0.$$ 

For any $u \in \mathcal{C}_A^0$, define $\hat{u}_\eta$ by:

$$\hat{u}_\eta(x) = \begin{cases} \eta u(\eta^{-1}(x - x_0) + x_0) + (1 - \eta)Ax_0, & x \in \Omega^\eta, \\ Ax, & \text{elsewhere}. \end{cases}$$

Clearly $\hat{u}_\eta \in \mathcal{C}_{\varepsilon}^A$. We now show that by adjusting $\eta$, the function $\hat{u}_\eta$ also satisfies the integral constraint in $\mathcal{C}_{\varepsilon}^A$. If we let $\Omega^\eta = \eta(\Omega - x_0) + x_0$, then

$$\int_{\Omega^\eta} \det \nabla \hat{u}_\eta \, dx = \int_{\Omega^\eta \setminus \partial \Omega^\eta(x_0)} \det \nabla \hat{u}_\eta \, dx + \int_{\Omega \setminus \Omega^\eta} \det \nabla \hat{u}_\eta \, dx = \eta^n \int_{\Omega \setminus \partial \Omega^\eta(x_0)} \det \nabla u \, dy + (\det A)(1 - \eta^n) |\Omega|,$$

where we used that $|\Omega^\eta| = \eta^n |\Omega|$. But

$$\int_{\Omega \setminus \partial \Omega^\eta(x_0)} \det \nabla u \, dy = \int_{\Omega} \det \nabla u \, dy - \int_{\Omega \setminus \partial \Omega^\eta(x_0)} \det \nabla u \, dy = (\det A) |\Omega| - V^\eta,$$

where

$$V^\eta = V + \int_{\Omega \setminus \partial \Omega^\eta(x_0)} \det \nabla u \, dy.$$
Using this we get now that
\[ \int_{\Omega_\varepsilon} \det \nabla \hat{u}_\eta \, dx = (\det A) |\Omega| - \eta^n V^\eta_{\varepsilon} . \]

Is easy to see that as a function of \( \eta \), the ratio \( V/V^\eta_{\varepsilon} \) is strictly increasing with
\[
\lim_{\eta \to 0} \frac{V}{V^\eta_{\varepsilon}} = \frac{V}{V_{\varepsilon}}, \quad \lim_{\eta \to 1} \frac{V}{V^\eta_{\varepsilon}} = \frac{V}{V_{\varepsilon}}.
\]

Since \( V_{\varepsilon} \searrow V \) as \( \varepsilon \searrow 0 \), we get that for \( \varepsilon \) sufficiently small, we can find \( \eta = \eta(\varepsilon) \in (0, 1) \) such that \( \eta^n V^\eta_{\varepsilon} = V \). (Note that \( \eta(\varepsilon) \nearrow 1 \) as \( \varepsilon \searrow 0 \).) With this choice of \( \eta(\varepsilon) \), we get that
\[
\int_{\Omega_\varepsilon} \det \nabla \hat{u}_{\eta(\varepsilon)} \, dy = (\det A) |\Omega| - V,
\]
concluding that \( \hat{u}_{\eta(\varepsilon)} \in \mathcal{A}_\varepsilon^\varepsilon \).

Henceforth we set \( \hat{u}_\varepsilon = \hat{u}_{\eta(\varepsilon)} \). We can extend \( \hat{u}_\varepsilon \) to \( \Omega \) by replacing \( \Omega_{\eta(\varepsilon)} \) in the definition of \( \hat{u}_{\eta(\varepsilon)} \). The resulting \( \hat{u}_\varepsilon \) now belongs to \( W^{1,p}(\Omega) \) with the additional property that \( \hat{u}_\varepsilon |_{\Omega_\varepsilon} \in \mathcal{A}_\varepsilon^\varepsilon \).

Now take \( \{\varepsilon_j\} \) with \( \varepsilon_j \to 0 \) and set \( \hat{u}_j = \hat{u}_{\varepsilon_j} \). It follows now that
\[
\int_\Omega W(\nabla \hat{u}_j) \, dx = \int_{\Omega_{\varepsilon_j}} W(\nabla \hat{u}_{\varepsilon_j}) \, dx + W(A) |\Omega \setminus \Omega_{\varepsilon_j}|
\]
\[
= \eta(\varepsilon_j)^n \int_\Omega W(\nabla u) \, dy + W(A) |\Omega \setminus \Omega_{\varepsilon_j}|
\]
\[
\to \int_\Omega W(\nabla u) \, dy,
\]
as \( j \to \infty \).

We now have one of the main results of this paper.

**Theorem 4** Let the hypotheses in Lemma 1 hold. For \( V \in (0, |\Omega| \det A) \), let \( \{\varepsilon_j\} \) be a sequence of positive numbers converging to zero, and for each \( \varepsilon_j \), let \( u_j \) be the corresponding minimizer given by Theorem 1 and satisfying (8). Then \( \{u_j\} \) has a subsequence \( \{u_{j_k}\} \) such that for any \( \delta > 0 \),
\[
u_{j_k} \to u_V \quad \text{in} \quad W^{1,p}(\Omega_\delta),
\]
where the function \( u_V \) is a solution of (6), and with
\[
E(u_V) = \lim_{k} E_{\varepsilon_{j_k}}(u_{j_k}). \tag{30}
\]

**Proof** We let \( \mathcal{A}_\varepsilon^\varepsilon \equiv \{u \in \mathcal{A}^\varepsilon | c_\varepsilon(u) = 0\}, \quad \varepsilon \geq 0. \)

It follows from Lemma 2 that these sets are non empty for \( \varepsilon \) small enough. Thus each \( u_j \) satisfies:
\[
E_{\varepsilon_j}(u_j) = \min_{u \in \mathcal{A}_\varepsilon^\varepsilon} \int_{\Omega_{\varepsilon_j}} W(\nabla u(x)) \, dx = \min_{u \in \mathcal{A}_\varepsilon^\varepsilon} E_{\varepsilon_j}(u).
\]
Now we fix an index $J \in \mathbb{N}$ and take $j > J$. It follows from hypothesis (H2) on $W$ and Poincaré’s inequality, that for some constants $K > 0$ and $L \in \mathbb{R}$:

$$E_{\varepsilon J}(u_j) \geq K \|u_j\|_{W^{1,p}(\Omega_{\varepsilon j})}^p + L, \quad j > J.$$ 

Again, it follows from (H2) that we may assume that $W$ is non-negative. Hence

$$E_{\varepsilon J}(u_j) \leq E_{\varepsilon j}(u_j) \leq C, \quad j > J,$$

where the constant $C$ is given by Lemma 2. Combining this with the previous inequality we get that (for a subsequence) $\{u_j\}$ converges weakly in $W^{1,p}(\Omega_{\varepsilon J})$ to a function $u'$, and that $\{\det \nabla u_j\}$ converges weakly in $L^1(\Omega_{\varepsilon J})$ to a function $\theta'$. Since $p \in (n-1,n)$, it follows from [14, Theorem 4.2], that $u'$ satisfies condition INV, $\theta' = \det \nabla u'$, and $\det \nabla u' > 0$ almost everywhere. By choosing an appropriate diagonal sequence, it is shown in [21] that there exists a subsequence $\{u_{jk}\}$ and a function $u_{V} \in W^{1,p}(\Omega)$ such that

$$u_{jk} \rightharpoonup u_{V}, \quad \text{in} \quad W^{1,p}(\Omega_{\varepsilon J}).$$

The results in [21, Section 4.2] show that $u_{V} \in \mathcal{A}_A$.

It remains to show that $u_{V}$ is a solution of (6). By the results quoted in the previous paragraph, we get that the subsequence $\{u_{jk}\}$ has the property that

$$\det \nabla u_{jk} \rightharpoonup \det \nabla u_{V}, \quad \text{in} \quad L^1(\Omega_{\varepsilon J}).$$

Since $u_{jk} \in \mathcal{A}_{\varepsilon J}$, we also have that

$$\int_{\Omega_{\varepsilon J}} \det \nabla u_{jk} \, dx = (\det A) |\Omega| - V, \quad \forall k.$$ 

Now we extend $\det \nabla u_{jk}$ to $\Omega$ as follows:

$$g_k(x) = \begin{cases} 
\det \nabla u_{jk}(x), & x \in \Omega_{\varepsilon J}, \\
0, & x \in \Omega \setminus \Omega_{\varepsilon J}.
\end{cases}$$

Clearly $g_k \in L^1(\Omega)$ and

$$\int_{\Omega} g_k \, dx = \int_{\Omega_{\varepsilon J}} \det \nabla u_{jk} \, dx = (\det A) |\Omega| - V, \quad \forall k.$$ 

Writing

$$\int_{\Omega} (\det \nabla u_V - g_k) \, dx = \int_{\Omega_{\varepsilon J}} (\det \nabla u_V - g_k) \, dx + \int_{\partial \Omega_{\varepsilon J}(x_0)} (\det \nabla u_V - g_k) \, dx,$$  (31)

we note that the second term above can be made arbitrarily small by taking $J$ sufficiently large. To see this we first observe that

$$\int_{\partial \Omega_{\varepsilon J}(x_0)} g_k \, dx = \int_{\partial \Omega_{\varepsilon J}(x_0)} \det \nabla u_{\varepsilon J} \, dx,$$
where $\mathcal{D}_k = \mathcal{B}_{\epsilon_k}(x_0) \setminus \mathcal{B}_{\epsilon_{jk}}(x_0)$. Now using Jensen’s inequality and the convexity of $h(\cdot)$, we get that

$$|\mathcal{D}_k| h\left(\frac{1}{|\mathcal{D}_k|} \int_{\mathcal{D}_k} \det \nabla u_{\epsilon_{jk}} \, dx\right) \leq \int_{\mathcal{D}_k} h(\det \nabla u_{\epsilon_{jk}}) \, dx.$$

By Lemma 2 in the Appendix, the right hand side of this inequality is uniformly bounded. Thus our statement about the second term in (31) now follows from (12b) and arguing by contradiction. Now once $J$ is fixed, the first term in (31) can be made arbitrarily small as $g_k$ equals $\det \nabla u_{\epsilon_{jk}}$ over $\Omega_{\epsilon_k}$, for $k$ sufficiently large and by the weak convergence of $\{\det \nabla u_{\epsilon_{jk}}\}$ to $\det \nabla u_V$ in $L^1(\Omega_{\epsilon_k})$. This shows that

$$\int_{\Omega} \det \nabla u_V \, dx = \lim_{k \to \infty} \int_{\Omega} g_k \, dx = (\det A) |\Omega| - V, \quad (32)$$

Hence $u_V \in \mathcal{C}^0_A$. We now show that $u_V$ is a minimizer over $\mathcal{C}^0_A$.

For any $u \in \mathcal{C}_A$ and for the subsequence $\{\epsilon_{jk}\}$ above, let $\hat{u}_{\epsilon_{jk}}$ be the corresponding sequence given by Lemma 1 with the property that

$$\lim_{k \to \infty} \int_{\Omega_{\epsilon_{jk}}} W(\nabla u_{\epsilon_{jk}}(x)) \, dx = \int_{\Omega} W(\nabla u(x)) \, dx. \quad (33)$$

As a function over $\Omega_{\epsilon_{jk}}$, we have that $\hat{u}_{\epsilon_{jk}} \in \mathcal{C}^\epsilon_{A_{\epsilon_{jk}}}$. Since $u_{\epsilon_{jk}}$ is the minimizer over $\mathcal{C}^\epsilon_{A_{\epsilon_{jk}}}$, we have that

$$\int_{\Omega_{\epsilon_{jk}}} W(\nabla u_{\epsilon_{jk}}(x)) \, dx \leq \int_{\Omega_{\epsilon_{jk}}} W(\nabla \hat{u}_{\epsilon_{jk}}(x)) \, dx. \quad (34)$$

Let $N > 0$ be given. For $k > N$ and the nonnegativity of $W$ we get that

$$\int_{\Omega_{\epsilon_{jk}}} W(\nabla u_{\epsilon_{jk}}(x)) \, dx \leq \int_{\Omega_{\epsilon_{jk}}} W(\nabla \hat{u}_{\epsilon_{jk}}(x)) \, dx. \quad (35)$$

By the results in [1], the functional $E_{\epsilon_{jk}}(\cdot)$ (cf. (8)) is weakly lower semi–continuous over $\mathcal{C}^{\epsilon_{jk}}_A$. Using this and since $u_{\epsilon_{jk}} \rightharpoonup u_V$ in $W^{1,p}(\Omega_{\epsilon_{jk}})$, we conclude that

$$\int_{\Omega_{\epsilon_{jk}}} W(\nabla u_V(x)) \, dx \leq \liminf_k \int_{\Omega_{\epsilon_{jk}}} W(\nabla u_{\epsilon_{jk}}(x)) \, dx. \quad (36)$$

From the nonnegativity of $W$, it follows from (33)–(36) that

$$\int_{\Omega_{\epsilon_{jk}}} W(\nabla u_V(x)) \, dx \leq \int_{\Omega} W(\nabla u(x)) \, dx.$$

Since $N$ is arbitrary, we can conclude that

$$\int_{\Omega} W(\nabla u_V(x)) \, dx \leq \int_{\Omega} W(\nabla u(x)) \, dx.$$
Since \( u \in \mathcal{C}_A^0 \) is arbitrary, we get that \( u_{\pm} \) is a minimizer over \( \mathcal{C}_A^0 \). If we set \( u = u_{\pm} \) in (33), then we get as well that
\[
\int_\Omega W(\nabla u_{\pm}(x)) \, dx = \liminf_k \int_{\Omega_{\epsilon_k}} W(\nabla u_{\pm}(x)) \, dx,
\]
from which the result about the energies follows upon taking another subsequence.

We now derive an expression for a weak form of the equilibrium equations for the minimizer \( u_{\pm} \) in Theorem 4. Although the next result seems similar to that of Theorem 2, the proof is somewhat more technical due to the fact that the domains of the sequence of approximating functions are changing with the sequential index.

**Theorem 5** Assume that (18) and the hypotheses in Theorem 4 hold, and that the stored energy function \( W \) is of the form \( \gamma |F|^p + \tilde{W}(F) \) where \( \gamma > 0 \) and \( \tilde{W} \) satisfy H1 and H3. Let \( u_{\pm} \) be the minimizer in Theorem 4. Then there exists \( \mu_{\pm} \in \mathbb{R} \) such that
\[
\int_\Omega \left[ \frac{dW}{dF}(\nabla u_{\pm}) + \mu_{\pm} (\text{adj} \, \nabla u_{\pm})^T \right] \cdot \nabla \left[ (\nabla u_{\pm}) \right] \, dx = 0,
\]
for all \( v \in C^1(\mathbb{R}^n) \) with \( v = 0 \) on \( \mathbb{R}^n \setminus \mathcal{E} \) where \( \mathcal{E} = \{Ax : x \in \Omega\} \). Moreover, if \( u_{\pm} \in C^2(\Omega \setminus \{x_0\}) \cap C^1(\Omega \setminus \{x_0\}) \) with \( \det \nabla u_{\pm} > 0 \) in \( \Omega \setminus \{x_0\} \), then
\[
\text{div} \left[ \frac{dW}{dF}(\nabla u_{\pm}) + \mu_{\pm} (\text{adj} \, \nabla u_{\pm})^T \right] = 0, \quad \text{in} \quad \Omega \setminus \{x_0\},
\]
and
\[
\lim_{\delta \to 0} \int_{\partial \Omega \setminus \{x_0\}} v(u_{\pm}) \cdot \left[ \frac{dW}{dF}(\nabla u_{\pm}) + \mu_{\pm} (\text{adj} \, \nabla u_{\pm})^T \right] \, n \, ds(x) = 0.
\]

**Proof** Let \( \{u_{\pm_k}\} \) be the subsequence given by Theorem 4 such that (30) holds and for any \( \delta > 0 \),
\[
u_{\pm_k} \rightharpoonup u_{\pm} \quad \text{in} \quad W^{1,p}(\Omega_\delta),
\]
\[
\det \nabla \nu_{\pm_k} \rightharpoonup \det \nabla u_{\pm} \quad \text{in} \quad L^1(\Omega_\delta).
\]

Here \( u_{\pm_k} \) is the minimizer given by Theorem 1 corresponding to \( \epsilon_{\pm_k} \). The proof is divided into several steps.

**Step 1:** We first show that
\[
\int_\Omega |\nabla u_{\pm}|^p \, dx = \lim_k \int_{\Omega_{\epsilon_{\pm_k}}} |\nabla u_{\pm_k}|^p \, dx.
\]
Note that
\[
\int_{\Omega_\delta} \tilde{W}(\nabla u_{\pm}) \, dx \leq \liminf_k \int_{\Omega_{\epsilon_{\pm_k}}} \tilde{W}(\nabla u_{\pm_k}) \, dx \leq \liminf_k \int_{\Omega_{\epsilon_{\pm_k}}} \tilde{W}(\nabla u_{\pm_k}) \, dx,
\]
from which it follows that
\[ \int_{\Omega} \mathcal{W}(\nabla u_V) \, dx \leq \liminf_{k} \int_{\Omega_{\varepsilon_{jk}}} \mathcal{W}(\nabla u_{jk}) \, dx. \]

Using this we have now:
\[
\int_{\Omega} W(\nabla u_V) \, dx - \gamma \int_{\Omega} |\nabla u_V|^p \, dx = \int_{\Omega} \mathcal{W}(\nabla u_V) \, dx \\
\leq \liminf_{k} \int_{\Omega_{\varepsilon_{jk}}} \mathcal{W}(\nabla u_{jk}) \, dx \\
\leq \lim_{k} \int_{\Omega_{\varepsilon_{jk}}} W(\nabla u_{jk}) \, dx - \gamma \limsup_{k} \int_{\Omega_{\varepsilon_{jk}}} |\nabla u_{jk}|^p \, dx,
\]
which upon invoking (30) yields that
\[
\limsup_{k} \int_{\Omega_{\varepsilon_{jk}}} |\nabla u_{jk}|^p \, dx \leq \int_{\Omega} |\nabla u_V|^p \, dx.
\]
Since \( u_{jk} \rightharpoonup u_V \) in \( W^{1,p}(\Omega_\delta) \), we get that
\[
\int_{\Omega_\delta} |\nabla u_V|^p \, dx \leq \liminf_{k} \int_{\Omega_\delta} |\nabla u_{jk}|^p \, dx \leq \liminf_{k} \int_{\Omega_{\varepsilon_{jk}}} |\nabla u_{jk}|^p \, dx,
\]
which by the arbitrariness of \( \delta \) leads to
\[
\int_{\Omega} |\nabla u_V|^p \, dx \leq \liminf_{k} \int_{\Omega_{\varepsilon_{jk}}} |\nabla u_{jk}|^p \, dx.
\]
This combined with our previous result yields (41).

**Step 2:** We now show that
\[ \nabla u_{jk} \rightharpoonup \nabla u_V \text{ a.e. in } \Omega_\delta, \]
for any \( \delta > 0 \). Let \( \psi : \mathbb{R} \to [0, \infty) \) be a smooth function such that \( \psi(t) = 0 \) for \( t \leq 1 \) and \( \psi(t) = 1 \) for \( t \geq \frac{4}{3} \). For each \( k \) let \( \tilde{u}_{jk} : \Omega \to \mathbb{R}^n \) be given by:
\[
\tilde{u}_{jk}(x) = \begin{cases} 
\psi \left( \frac{2|x - x_0|}{|x - x_0| + \varepsilon_{jk}} \right) u_{jk}(x), & x \in \Omega_{\varepsilon_{jk}}, \\
0, & x \in \Omega \setminus \Omega_{\varepsilon_{jk}}.
\end{cases}
\]

Using (40a) one can show now that \( \tilde{u}_{jk} \rightharpoonup u_V \) in \( W^{1,p}(\Omega) \). Furthermore, we now show that
\[
\int_{\Omega} |\nabla u_V|^p \, dx = \lim_{k} \int_{\Omega} |\nabla \tilde{u}_{jk}|^p \, dx. \tag{42}
\]
We shall need to estimate the \( L_p \) norms of the gradients of both \( u_{jk}, \tilde{u}_{jk} \) on the strip
\[ E_k = \Omega_{\varepsilon_{jk}} \setminus \Omega_{2\varepsilon_{jk}} = B_{2\varepsilon_{jk}}(x_0) \setminus B_{\varepsilon_{jk}}(x_0). \]
To do this, first note that using (41) and the weak lower semi–continuity of the $L^p(\Omega)$ norm, one can easily show that
\[ \int_{\Omega} |\nabla u_V|^p \, dx = \lim_{\delta \to 0} \left[ \liminf_{k} \int_{\Omega^{\delta}} |\nabla u_k|^p \, dx \right]. \] (43)
This together with (41) once again can be used to show that up to a subsequence (not relabeled) of \{u_k\}, there exists \{\delta_k\} with \delta_k \to 0 and $2\epsilon_k < \delta_k$, such that
\[ \int_{\Omega} |\nabla u_V|^p \, dx = \lim_{k \to \infty} \int_{\Omega^{\delta_k}} |\nabla u_k|^p \, dx. \]
Now
\[ \int_{\Omega^{\delta_k}} |\nabla u_k|^p \, dx = \int_{\Omega^{\delta_k}} |\nabla u_A|^p \, dx + \int_{\Omega^{\delta_k} \setminus \Omega^{\delta_k}} |\nabla u_k|^p \, dx \]
Since both, the left hand side of this equation and the first term on the right, converge to $\int_{\Omega} |\nabla u_V|^p \, dx$, we get that
\[ \int_{\Omega^{\delta_k} \setminus \Omega^{\delta_k}} |\nabla u_k|^p \, dx \to 0, \quad k \to \infty. \]
Since $2\epsilon_k < \delta_k$, we have that
\[ \int_{E_k} |\nabla u_k|^p \, dx \leq \int_{\Omega^{\delta_k} \setminus \Omega^{\delta_k}} |\nabla u_k|^p \, dx \to 0, \quad k \to \infty. \] (44)
In addition we have, for some constant $C$ depending on $\psi$, that
\[ \int_{E_k} |\nabla \tilde{u}_k|^p \, dx \leq C \left[ \int_{E_k} |\nabla \tilde{u}_k|^p \, dx + \frac{|E_k|}{(2\epsilon_k)^p} \int_{\Omega^{\delta_k}} |\nabla u_k|^p \, dx \right] \to 0, \] (45)
as $|E_k| = \frac{\omega_n}{2^n} (2^n - 1) \epsilon_k^n$ with $p < n$, and by (44). Now for the $L_p(\Omega)$ norm of the gradient of $\tilde{u}_k$, we have that
\[ \int_{\Omega} |\nabla \tilde{u}_k|^p \, dx = \int_{E_k} |\nabla \tilde{u}_k|^p \, dx + \int_{\Omega^{\delta_k} \setminus \Omega^{\delta_k}} |\nabla \tilde{u}_k|^p \, dx, \] (46)
Now for the second term to the right of this equation we have that
\[ \int_{\Omega^{\delta_k} \setminus \Omega^{\delta_k}} |\nabla \tilde{u}_k|^p \, dx = \int_{\Omega^{\delta_k} \setminus \Omega^{\delta_k}} |\nabla u_k|^p \, dx - \int_{E_k} |\nabla \tilde{u}_k|^p \, dx \]
\[ \to \int_{\Omega} |\nabla u_V|^p \, dx, \] (47)
by (44) and (41). Thus combining (45)–(47), we get that (42) holds. Since $\tilde{u}_k \to u_V$ in $W^{1,p}(\Omega)$, it follows now that (for a subsequence) $\nabla \tilde{u}_k \to \nabla u_V$ a.e. in $\Omega$. Since $\tilde{u}_k(x) = u_k(x)$ for all $x \in \Omega^{\delta_k}$, it follows that $\nabla u_k \to \nabla u_V$ a.e. in $\Omega^{\delta}$ for any $\delta > 0$.

**Step 3:** Finally we show now that the weak form (37) of the equilibrium equations
for the minimizer $u_V$ holds. Once again the varying domains in the sequence $\{u_{j_k}\}$ complicates the analysis. The convergence a.e. established in Step 2 is an essential ingredient for the following arguments.

Let $N > 0$ be such that $\varepsilon_{j_k} < \delta$ for $k > N$. Using Theorem 2 we get now that

$$0 = \int_{\Omega_{\varepsilon_{j_k}}} \left[ \frac{dW}{dF}(\nabla u_{j_k}) \nabla u_{j_k}^T + \mu_{j_k}(\det \nabla u_{j_k}) I \right] \cdot \nabla v(u_{j_k}) \, dx$$

$$= \int_{\Omega_{\delta}(x_0) \setminus \Omega_{\varepsilon_{j_k}(x_0)}} \left[ \frac{dW}{dF}(\nabla u_{j_k}) \nabla u_{j_k}^T + \mu_{j_k}(\det \nabla u_{j_k}) I \right] \cdot \nabla v(u_{j_k}) \, dx$$

$$+ \int_{\Omega_{\delta}} \left[ \frac{dW}{dF}(\nabla u_{j_k}) \nabla u_{j_k}^T + \mu_{j_k}(\det \nabla u_{j_k}) I \right] \cdot \nabla v(u_{j_k}) \, dx, \quad k > N. \quad (48)$$

for all $v \in C^1(\mathbb{R}^n)$ with $v = 0$ on $\mathbb{R}^n \setminus \delta$. It follows from Step 2, hypothesis (18) and the generalized Dominated Convergence Theorem (see [19]), that

$$\lim_k \int_{\Omega_{\delta}} \left[ \frac{dW}{dF}(\nabla u_{j_k}) \nabla u_{j_k}^T \right] \cdot \nabla v(u_{j_k}) \, dx = \int_{\Omega_{\delta}} \left[ \frac{dW}{dF}(\nabla u_V) \nabla u_V^T \right] \cdot \nabla v(u_V) \, dx.$$

Also from (40b) and [20, Lemma 6.7] we get that

$$\lim_k \int_{\Omega_{\delta}} (\det \nabla u_{j_k}) I \cdot \nabla v(u_{j_k}) \, dx = \int_{\Omega_{\delta}} (\det \nabla u_V) I \cdot \nabla v(u_V) \, dx.$$

By and argument similar to one within the proof of Theorem 4 (cf. (31)), we get that the integrals

$$\int_{\Omega_{\delta}(x_0) \setminus \Omega_{\varepsilon_{j_k}(x_0)}} (\det \nabla u_{j_k}) I \cdot \nabla v(u_{j_k}) \, dx,$$

can be made arbitrarily small as $\delta \downarrow 0$ and $k \to \infty$. Now let

$$w_k(x) = \begin{cases} W(\nabla u_{j_k}(x)), & x \in \Omega_{\varepsilon_{j_k}}; \\ 0, & \text{elsewhere} \end{cases}$$

Since $\nabla u_{j_k} \to \nabla u_V$ a.e. in $\Omega_{\delta}$ for any $\delta > 0$, we have that $w_k \to W(\nabla u_V)$ a.e. in $\Omega$. Also

$$\|w_k\|_{L^1(\Omega)} = \int_{\Omega_{\varepsilon_{j_k}}} W(\nabla u_{j_k}(x)) \, dx \to \int_{\Omega} W(\nabla u_V(x)) \, dx = \|W(\nabla u_V)\|_{L^1(\Omega)},$$

by (30). It follows now that $w_k \to W(\nabla u_V)$ in $L^1(\Omega)$, which implies that $\{w_k\}$ is equi-integrable. This property of the $w_k$'s together with (18) can be used now to show that the integrals

$$\int_{\Omega_{\delta}(x_0) \setminus \Omega_{\varepsilon_{j_k}(x_0)}} \frac{dW}{dF}(\nabla u_{j_k}) \nabla u_{j_k}^T \cdot \nabla v(u_{j_k}) \, dx,$$

can be made arbitrarily small as $\delta \downarrow 0$ and $k \to \infty$. Thus letting first $k \to \infty$ in (48), and then letting $\delta \downarrow 0$, we get that for some $\mu_V \in \mathbb{R}$,

$$\int_{\Omega} \left[ \frac{dW}{dF}(\nabla u_V) \nabla u_V^T + \mu_V(\det \nabla u_V) I \right] \cdot \nabla v(u_V) \, dx = 0,$$
for all $v \in C^1(\mathbb{R}^n)$ with $v = 0$ on $\mathbb{R}^n \setminus \delta$.

Now assume that $u_V \in C^2(\Omega \setminus \{x_0\}) \cap C^1(\overline{\Omega} \setminus \{x_0\})$ with $\det \nabla u_V > 0$ in $\Omega \setminus \{x_0\}$. The proof of (38) is similar to the one given in [20, Theorem 5.1] and thus we omit it. Let $\delta > 0$ be given. If we multiply (38) by $v(u_V)$, where $v \in C^1(\mathbb{R}^n)$ with $v = 0$ on $\mathbb{R}^n \setminus \delta$ and constant over $\mathcal{H}$, and integrate by parts over $\Omega_\delta$, we get that

$$
\int_{\Omega_\delta} \left[ \frac{dW}{dF}(\nabla u_V) + \mu_V (\operatorname{adj} \nabla u_V) \right]^T \nabla v(u_V) \, dx = \int_{\partial \Omega_\delta(x_0)} v(u_V) \cdot \left[ \frac{dW}{dF}(\nabla u_V) + \mu_V (\operatorname{adj} \nabla u_V) \right] n \, ds(x).
$$

Taking the limit as $\delta \searrow 0$ and using (37) we get that (39) holds. \qed

Remark 5 In terms of the Cauchy stress tensor:

$$
T(u_V) = (\det \nabla u_V)^{-1} \frac{dW}{dF}(\nabla u_V)(\nabla u_V)^T,
$$

(39) is equivalent to:

$$
\lim_{\delta \to 0} \int_{u_V(\partial \Omega_\delta(x_0))} v(y) \cdot [T(y) + \mu_V I \hat{n}(y)] \, ds(y) = 0,
$$

where $\hat{n}$ is the unit normal to $u_V(\partial \Omega_\delta(x_0))$. If $H$ is the region of volume $V$ occupied by the cavity induced by $u_V$, then the limit above can be replaced by the corresponding integral over $\partial H$. It follows now that

$$
\int_{\partial H} v(y) \cdot [T(y) + \mu_V I \hat{n}(y)] \, ds(y) = 0,
$$

for all $v \in C^1(\mathbb{R}^n)$. Thus

$$
[T(y) + \mu_V I] \cdot \hat{n}(y) = 0, \text{ over } \partial H,
$$

in the sense of trace.

We now establish a very nice connection between the Lagrange multipliers $\mu_V$ and the volume derivative (cf [17]). The volume derivative of the stored energy function $W$ at the boundary displacement $A$ is given by:

$$
G(A) = \lim_{V \to 0^+} \inf_{u \in \mathcal{A}_V} \frac{E(u) - E(A)}{V} = \lim_{V \to 0^+} \frac{E(u_V) - E(A)}{V},
$$

where $E(\cdot)$ is as in (1) and $u_V$ is the minimizer from Theorems 4 and 5. In the following, we write $u(x,V)$ instead of $u_V(x)$.

**Theorem 6** Assume that (18) and the hypotheses in Theorems 4 and 5 hold and that:

1. $u(\cdot,V) \in C^2(\Omega \setminus \{x_0\}) \cap C^1(\overline{\Omega} \setminus \{x_0\})$ with $\det \nabla u_V > 0$ in $\Omega \setminus \{x_0\}$;
2. $u(\cdot,V) \in C^1(\Omega \setminus \{x_0\}) \times (0,V_0)$ for some $V_0 > 0$;
3. for some $\delta > 0$, $u(\cdot,V) \to u_h(\cdot)$ in $C^2(\overline{\Omega} \setminus \partial \delta(x_0))$ as $V \searrow 0$, where $u_h(x) = Ax$ for all $x \in \overline{\Omega}$. 


Then
\[ G(A) = \lim_{V \to 0^+} \mu_V, \]
where \( \{ \mu_V \} \) are the Lagrange multipliers from Theorem 5.

**Proof** Under the stated hypotheses it is shown in [17, Prop. 5.4] that
\[
G(A) = -\frac{1}{n} \lim_{\delta \to 0^+} \frac{1}{V} \left[ \lim_{\delta \to 0^+} \int_{\partial \mathcal{B}_\delta(x_0)} (u_V - Ax_0) \cdot \frac{\partial W}{\partial F}(\nabla u_V) n ds(x) \right],
\]
where the normal \( n \) to \( \partial \mathcal{B}_\delta(x_0) \) points in the outward direction. Let \( K \subset \Omega \) be a compact set such that \( u_V(\mathcal{B}_\delta(x_0)) \subset K \) for all \( \delta \) and \( V \) sufficiently small. Then taking \( v \) in (39) such that \( v(x) = x - Ax_0 \) for \( x \in K \), we get that
\[
\lim_{\delta \to 0^+} \int_{\partial \mathcal{B}_\delta(x_0)} (u_V - Ax_0) \cdot (\text{adj} \nabla u_V)^T n ds(x).
\]
But
\[
\int_{\partial \mathcal{B}_\delta(x_0)} (u_V - Ax_0) \cdot (\text{adj} \nabla u_V)^T n ds(x)
= \int_{\partial \Omega} (u_V - Ax_0) \cdot (\text{adj} \nabla u_V)^T n ds(x) - n \int_{\Omega \setminus \mathcal{B}_\delta(x_0)} \det \nabla u_V dx.
\]
Thus
\[
\lim_{\delta \to 0^+} \int_{\partial \mathcal{B}_\delta(x_0)} (u_V - Ax_0) \cdot (\text{adj} \nabla u_V)^T n ds(x)
= \int_{\partial \Omega} (u_V - Ax_0) \cdot (\text{adj} \nabla u_V)^T n ds(x) - n \int_{\Omega} \det \nabla u_V dx
= n(\det A)|\Omega| - n[(\det A)|\Omega| - V] = nV.
\]
It follows now that
\[
G(A) = -\frac{1}{n} \lim_{V \to 0^+} \frac{1}{V} (-\mu_V nV) = \lim_{V \to 0^+} \mu_V.
\]
\[\square\]

4 **Numerical results**

In this section we describe some of the elements of a numerical procedure, based on the results of the previous sections, to compute a minimizer of (6). In addition we work a numerical example in which we check the convergence as \( \varepsilon \to 0 \) predicted by Theorem 4 and another example in which we test the robustness of the method in the so called incompressible limit.
For given values of $\varepsilon, V$, we use the method outlined in Theorem 1 to compute the minimizer $u_\varepsilon$ in (8). The minimizers in (15a) (dropping the subscript “$j$”) are computed using the gradient flow equation:

$$\Delta u = -\text{div} \left[ \frac{dW}{dF}(\nabla u) + (\mu + \eta c_{\varepsilon}(u))(\text{adj} \nabla u)^T \right], \text{ in } \Omega_{\varepsilon},$$  \hspace{1cm} (51)

where for all $t \geq 0$, $u(x,t) = Ax$ over $\partial \Omega$ and

$$\left[ \nabla u_t + \frac{dW}{dF}(\nabla u) + (\mu + \eta c_{\varepsilon}(u))(\text{adj} \nabla u)^T \right] n = 0, \text{ on } \partial \mathcal{B}_{\varepsilon}(x_0).$$  \hspace{1cm} (52)

The gradient flow equation leads to a descent method for the solution of (15a). (For more details about the gradient flow method and its properties we refer to [18], and for its use in problems leading to cavitation see [7].) After discretization of the partial derivative with respect to “$t$”, one can use a finite element method to solve the resulting flow equation. In particular, if we let $\Delta t > 0$ be given, and set $t_{i+1} = t_i + \Delta t$ where $t_0 = 0$, we can approximate $u_t(x,t_i)$ with:

$$z_i(x) = \frac{u_{i+1}(x) - u_i(x)}{\Delta t},$$

where $u_i(x) = u(x,t_i)$, etc. (We take $u_0(x)$ to be some initial deformation satisfying the boundary condition on $\partial \Omega$, e.g., $Ax$.) Inserting this approximation into the weak form of (51), (52), and evaluating the right hand side of (51) at $u = u_i$, we arrive at the following iterative formula:

$$\int_{\Omega_{\varepsilon}} \nabla z_i \cdot \nabla v \, dx + \int_{\Omega_{\varepsilon}} \left[ \frac{dW}{dF}(\nabla u_i) + (\mu + \eta c_{\varepsilon}(u_i))(\text{adj} \nabla u_i)^T \right] \cdot \nabla v \, dx = 0,$$  \hspace{1cm} (53)

for all $v$ vanishing on $\partial \Omega$ and sufficiently smooth so that the integrals above are well defined. Given $u_i$, one can solve the above equation for $z_i$ via some finite element method, and then set $u_{i+1} = u_i + \Delta t z_i$. This process is repeated for $i = 0, 1, \ldots$, until $u_{i+1} - u_i$ is “small” enough ($10^{-3}$ in the calculations below), or some maximum value of “$t$” is reached, declaring the last $u_i$ as an approximation of $u_\varepsilon$. This whole process is repeated for smaller values of of $\varepsilon$, to obtain as a result an approximation of the minimizer $u_V$ in (6).

For the computations we used the stored energy function (11) in which:

$$h(d) = c_1 d^{e_1} + c_2 d^{-e_2},$$

where $c_1, c_2 \geq 0$ and $e_1, e_2 > 0$. The reference configuration is stress free provided:

$$c_2 = \frac{\mu (\sqrt{\pi})^{-2} + c_1 e_1}{e_2}.$$

The case $\kappa = 0$ in (11) is called an elastic fluid.

---

\textsuperscript{4} It follows from (22) that the Euler–Lagrange equations for (15a) are formally given by equating to zero the right hand side of (51).
For an elastic fluid in which \( \Omega = \mathcal{B} \equiv \mathcal{B}_1(0) \) and \( x_0 = 0 \), the minimizer \( u_V \) in (6) is given by (see [17]):

\[
\begin{align*}
  u_V(x) &= \left[ d R^n + (1 - d) \right]^{1/n} \frac{Ax}{R}, \quad R = \|x\|,
\end{align*}
\]

where \( d \) is given by

\[
  d = 1 - \frac{nV}{\omega_n \det A}.
\]

(\( V \) is assumed to be sufficiently small as to guarantee that \( d > 0 \).) It follows that \( \det \nabla u_V = d \det A \). Thus we have that

\[
\begin{align*}
  E(u_V) &= \int_{\mathcal{B}} h(\det \nabla u_V) \, dx = \frac{\omega_n}{n} h(d \det A),
\end{align*}
\]

where \( \omega_n \) denotes the area of the unit sphere in \( \mathbb{R}^n \). We now consider the particular case in which

\[
  n = 2, \quad c_1 = 1, \quad e_1 = 2, \quad e_2 = 1, \quad V = \pi(0.15)^2, \quad A = \text{diag}(1.1, 1.4). \tag{54}
\]

Using the formulas above, we get that

\[
\begin{align*}
  E(u_V) &= \pi h((1.1)(1.4) - 0.15^2) = 11.3750.
\end{align*}
\]

For the parameters in Theorem 1 we used \( \gamma = 0.25, \beta = 2 \), with the stopping criteria in (15) given by \( |\mu_{j+1} - \mu_j| < 10^{-3} |\mu_j| \). The solution of the sub–problems (53) was done using the package freefem++ (see [4]) with first-order Crouzeix–Raviart finite elements. We show in Table 1 the results in this case for the method described at the beginning of this section for the data (54). Each line in this table shows, for a given \( \varepsilon \), the last computed step of the method outlined in Theorem 1. Note that the penalty parameter \( \eta \) (fifth column) do not become too large, thus avoiding the ill–conditioning associated with large values of these parameters. Also the computed energy values are approaching the exact energy 11.3750 in accordance with the result in Theorem 4, to within the convergence tolerances in the gradient flow and penalty multiplier iterations, and finite element approximations. In Figure 1 we show the initial finite element mesh and final computed deformation corresponding to \( \varepsilon = 0.00625 \). The hole (which is not circular) inside the computed deformation satisfies the constraint of having area \( V = \pi(0.15)^2 \) with an error of the order of \( O(10^{-7}) \).

For the stored energy corresponding to an elastic fluid, it is shown in [17] that the volume derivative at the matrix \( A \) is given by \( -h'(\det A) \). For the data (54) we get a value of \(-2.2367\) for the volume derivative. If we repeat the calculation in Table 1 corresponding to \( \varepsilon = 0.00625 \) but with prescribed area \( V = \pi(0.01)^2 \), we get a multiplier value upon convergence of \(-2.2375\), which approximates quite well the exact volume derivative to within the convergence tolerances.

The incompressible case of our problem corresponds to the case in which \( d \) is set to one in the \( h \) term of (11), and we minimize in (6) subject to the additional

\[
\begin{align*}
  \text{The minimizer in (8) is given by a similar expression but replacing } d \text{ with } d_\varepsilon = d/(1 - \varepsilon^n).
\end{align*}
\]
Table 1: Convergence of the regularized penalty–multiplier minimizers for the case of a two dimensional elastic fluid and data (54).

<table>
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<th>ε</th>
<th>c((u_\epsilon))</th>
<th>E((u_\epsilon))</th>
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<th>(\eta)</th>
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<tr>
<td>0.00625</td>
<td>4.99878e-06</td>
<td>11.3723</td>
<td>-2.17622</td>
<td>160</td>
</tr>
</tbody>
</table>

Fig. 1: Finite element mesh and computed minimizer via the penalty–multiplier method for the case of an elastic fluid and a spherical domain for the data (54).

constraint of \(\det \nabla \mathbf{u} = 1\). In this case, for \(\Omega = \mathcal{B}_1(0)\) and \(\mathbf{A} = \lambda \mathbf{I}\), the minimizer \(u_\lambda\) must be radial, that is \(u_\lambda(\mathbf{x}) = r(\|\mathbf{x}\|)\frac{x}{\|\mathbf{x}\|}\), where

\[
r(R) = \sqrt{R^2 + \lambda^n - 1}, \quad \lambda^n = 1 + c_\nu^n, \quad V = \frac{\alpha_n}{n} c_\nu^n.
\]  

(Note that the boundary displacement \(\lambda\) is completely determined by the volume constraint parameter \(V\)). Our next simulation is for what is called the incompressible limit. In particular, we consider the stored energy function (11) in which

\[
h(d) = c_1 d^{e_1} + c_2 d^{-e_2} + k(d - 1)^2,
\]  

where \(c_1, c_2 \geq 0, e_1, e_2 > 0,\) and \(k \geq 0\) is a “large” parameter. The parameters for the simulations (not including \(k\)) where taken to be:

\[n = 2, \quad \kappa = 1, \quad q = 1.5, \quad c_1 = 1, \quad e_1 = 2, \quad e_2 = 1, \quad c_\nu = 0.5.\]

The energy of the discrete version of (55) is given approximately by 16.6089. In Table 2 we show the results obtained by solving the nearly incompressible problem (8) (using (56) in (11)) via the regularized penalty–multiplier method (with \(\epsilon = 0.05\)) for increasing values of \(k\). The results in columns three and four in Table 2 show that
A penalty-multiplier method for cavitation with prescribed cavity size

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$E(u_\varepsilon)$</th>
<th>$E_2(u_\varepsilon)$</th>
<th>$|\det \nabla u_\varepsilon - 1|_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>9.71729e-05</td>
<td>16.6068</td>
<td>6.8198e-03</td>
</tr>
<tr>
<td>100</td>
<td>1.53148e-05</td>
<td>16.6074</td>
<td>9.20077e-04</td>
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<tr>
<td>1000</td>
<td>6.51531e-07</td>
<td>16.6075</td>
<td>9.47614e-05</td>
</tr>
</tbody>
</table>

Table 2: Results obtained using the penalty–multiplier method in the incompressible limit case.

both, the energy of the incompressible exact solution (55) and the incompressibility condition, are approximated quite well to within the discretization and convergence tolerances. In Figure 2 we show a graph of the determinant of the computed approximate minimizer corresponding to the last line in Table 2.

5 Final Comments

In [17, Proposition 6.1] the authors introduced a regularized penalty method for approximating solutions of (6). They anticipated without proof, the convergence of the corresponding regularised minimizers to a solution of (6) as $\varepsilon \searrow 0$. The result in Theorem 4 fills that gap and by using a penalty-multiplier scheme, we are led to a more stable numerical scheme as compared to the standard penalty method. This is the case as in general one achieves convergence in the penalty–multiplier method without having to make the penalty parameter excessively large, which could lead to numerical ill conditioning. We should point out that since the minimizers in (6) are not necessarily unique, the results of Theorem 4 are true for one such minimizer and the convergence is for a subsequence.
For the following discussion we employ the notation $\mu_\varepsilon(A, V)$ for the multiplier in Theorems 2 and 3. Let

$$\mathcal{F}_\varepsilon = \{ A : \mu_\varepsilon(A, V) = 0, A = \text{diag}(\lambda_1, \ldots, \lambda_n), \lambda_i > 0, 1 \leq i \leq n \}.$$  \hspace{1cm} (57)

Note that for the matrices in $\mathcal{F}_\varepsilon$, the corresponding minimizer $u_\varepsilon$ produces a stress-free inner cavity of volume $V$ (cf. 20c). Assume now that $\Omega = B_1(0)$ and that $x_0 = 0$. Given $V, \varepsilon > 0$ small enough, there exists a $\hat{\lambda} > 0$ (depending on $V$ and $\varepsilon$) and a radial minimizer $u_\varepsilon^c$ (cf. [16]) such that $\mu_\varepsilon(\hat{\lambda}, V) = 0$. Assuming this radial minimizer is also a global minimizer on $\mathcal{F}_\varepsilon^c$, equation (25b) reduces to:

$$\frac{\partial \mu_\varepsilon}{\partial \lambda_i} \bigg|_{\lambda_i} = \int_{\partial \Omega} x_i e_i \cdot \mathbf{C}(\nabla u_\varepsilon^c) [\mathbf{V} u_\varepsilon^c] \mathbf{n} dx + \frac{\partial \mu_\varepsilon}{\partial V} \left[ \int_{\partial \Omega} x_i e_i \cdot (\text{adj} \mathbf{V} u_\varepsilon^c)^T \mathbf{n} dx - \frac{\omega_n \hat{\lambda}^{n-1}}{n} \right].$$ \hspace{1cm} (58)

We now show that the integral $\int_{\partial \Omega} x_i e_i \cdot (\text{adj} \mathbf{V} u_\varepsilon^c)^T \mathbf{n} dx$ is independent of $i$. First recall that if $u_\varepsilon^c(x) = r_\varepsilon(R) \frac{x}{R}$, $R = ||x||$, then

$$\text{adj} \mathbf{V} u_\varepsilon^c(x) = \left[ r_\varepsilon(R) \frac{x}{R} \otimes \frac{x}{R} + r'_\varepsilon(R) \left( \frac{r_\varepsilon(R)}{R} \right)^{n-2} \left( 1 - \frac{x}{R} \otimes \frac{x}{R} \right) \right].$$

Let $Q$ be an orthogonal matrix such that $e_1 = Q e_i$, and let $y = Qx$ be a change of variables. It follows now that

$$Q (\text{adj} \mathbf{V} u_\varepsilon^c(x))^T Q^T = (\text{adj} \mathbf{V} u_\varepsilon^c(y))^T,$$

and that $y_1 = y \cdot e_1 = x \cdot Q^T e_1 = x \cdot e_i = x_i$. Thus

$$\int_{\partial \Omega} x_i e_i \cdot (\text{adj} \mathbf{V} u_\varepsilon^c(x))^T \mathbf{n}(x) ds(x) = \int_{\partial \Omega} x_i e_i \cdot Q (\text{adj} \mathbf{V} u_\varepsilon^c(x))^T Q^T \mathbf{n}(x) ds(x)$$

$$= \int_{Q(\partial \Omega)} y_1 e_1 \cdot (\text{adj} \mathbf{V} u_\varepsilon^c(y))^T \mathbf{n}(y) ds(y)$$

$$= \int_{\partial \Omega} y_1 e_1 \cdot (\text{adj} \mathbf{V} u_\varepsilon^c(y))^T \mathbf{n}(y) ds(y)$$

It follows now that

$$\int_{\partial \Omega} x_i e_i \cdot (\text{adj} \mathbf{V} u_\varepsilon^c)^T \mathbf{n} dx = \frac{1}{\omega_n} \int_{\partial \Omega} u_\varepsilon^c \cdot (\text{adj} \mathbf{V} u_\varepsilon^c)^T \mathbf{n} dx = \frac{\omega_n}{n} \hat{\lambda}^{n-1},$$

and that (58) simplifies to

$$\frac{\partial \mu_\varepsilon}{\partial \lambda_i} \bigg|_{\lambda_i} = \int_{\partial \Omega} x_i e_i \cdot C(\nabla u_\varepsilon^c) [\mathbf{V} u_\varepsilon^c] \mathbf{n} ds.$$

\hspace{1cm} (The minimizer $u_\varepsilon^c$ is a solution to a shell problem and is not the restriction to $\Omega_\varepsilon = B_1(0) \setminus \mathcal{F}_\varepsilon(0)$ of the minimizer over the solid ball.)
Note that \( C(\nabla u'_e)[\nabla u'_e] = \frac{\partial}{\partial V} \left[ \frac{dW}{dF}(\nabla u'_e) \right] \),
where the arguments of \( \Phi_1, \Phi_2 \) are \( (r'_e(R), r_e(R)/R, \ldots, r_e(R)/R) \). Thus using a symmetry argument similar to the one above, we get that
\[
\frac{\partial \mu_e}{\partial \lambda} \bigg|_{\lambda = \hat{\lambda}} = \frac{1}{n} \int_{\partial \Omega} n \cdot C(\nabla u'_e)[\nabla u'_e] n \, ds = \frac{\omega_e}{n} \frac{\partial \lambda}{\partial \lambda} \Phi_1(r'_e(1), \lambda, \ldots, \lambda) \bigg|_{\lambda = \hat{\lambda}},
\]
for \( i = 1, \ldots, n \). Now from [16, Page 707] we have that \( \frac{\partial \lambda}{\partial \lambda} > 0 \). Moreover from [1, Eq. (7.41)] we get that generically, the derivative with respect to \( \lambda \) of \( \Phi_1(r'_e(1), \lambda, \ldots, \lambda) \) is nonzero. (This is the case when \( k = 0 \) in (11).) Thus for those values of \( \hat{\lambda} \) for which this derivative is nonzero, we can invoke the Implicit Function Theorem to get that \( F^c_\hat{\lambda} \) is an \( n-1 \) dimensional surface in the neighbourhood of \( A = \hat{\lambda}I \). The matrices in this neighbourhood would each lead to solutions of (8) that produce a stress-free cavity or hole, not necessarily spherical, of volume \( V \). Thus the computation of \( F^c_\hat{\lambda} \) for progressively smaller values of \( V, \epsilon \), in a certain sense generalizes to the nonradial case the inverse method proposed in [16] for computing the critical \( \lambda \) in the radial case.

The set (cf. (50)):
\[
\mathcal{F} = \{ A \in M^{n \times n}_+ : G(A) = 0 \},
\]
is called the fracture surface associated to the stored energy function \( W \). In [17] the authors give justifications for the interpretation of \( \mathcal{F} \) as the boundary of the set of boundary displacement matrices leading or inducing to cavitation as defined in [1]. By combining Theorems 2 and 6 we get that
\[
G(A) = \lim_{V \to 0} \lim_{\epsilon \to 0} \mu_e(A, V).
\]
Thus the computation of the sets \( F^c_\lambda \) in (57) leads to a numerical scheme for approximating the fracture surface \( \mathcal{F} \), perhaps more robust than the one employed in [17] based on difference quotients.

### A The constrained admissible set

In this section we show that for \( \epsilon \) sufficiently small, the admissible sets in (8) are non–empty. In the proof we make use of the following result. let \( \mathcal{B}_1(0) \) be the unit ball with centre at the origin and for any \( d \in (0, 1) \), define
\[
u_d(x) = (dR^e + (1 - d)G) \frac{x}{R}, \quad R = \|x\|, \quad x \in \mathcal{B}_1(0).
\]
It follows from [1, Lemma 4.1], that \( \nu_d \in W^{1,p}(\mathcal{B}_1(0)) \) for \( p \in [1, n) \). An easy calculation shows as well that \( \det \nabla \nu_d = d \).

---

7 Here \( W(F) = \Phi(v_1, \ldots, v_n) \) where \( \Phi \) is a symmetric function and \( v_1, \ldots, v_n \) are the singular values of \( F \).
Lemma 2 There exists $V_0 > 0$ such that for any $V \in (0, V_0)$, there exists $\epsilon_0(V) > 0$ such that

$$c^\epsilon_0 \equiv \{ u \in \mathcal{E}^\epsilon \mid c_\epsilon(u) = 0 \} \neq \emptyset,$$

for all $\epsilon \in [0, \epsilon_0(V))$. Moreover, if $W$ is nonnegative and for any $0 < \gamma < \delta$ there exists a constant $K > 0$ such that

$$W(F) \leq K(\|F\|^p + 1), \quad \text{whenever } \det F \in [\gamma, \delta],$$

(60)

then for any nonnegative sequence $\epsilon_j \to 0$, there exists a sequence $z_{\epsilon_j} \in \mathcal{E}^\epsilon_{\epsilon_j}$ such that

$$E_{\epsilon_j}(z_{\epsilon_j}) \leq C, \quad \forall j,$$

for some constant $C > 0$.

Proof Let $\eta > 0$ be such that $\partial \eta(x_0) \subset \Omega$. Set $V_0 = \eta^n(a_0/n) \det A$, and let

$$\epsilon_0^\eta(V) = \frac{nV}{\det A \eta} = \eta^n \frac{V}{V_0}, \quad 0 < V < V_0,$$

$$c_\epsilon = \frac{\eta^n - \epsilon_0^n(V)}{\eta^n}, \quad \epsilon \in [0, \epsilon_0(V)).$$

It follows now that $c_\epsilon \in (0, 1)$ and that $B_{c_\epsilon}(x_0) \subset \partial \eta(x_0)$. Now define $w_\epsilon = u_{d_\epsilon}$ and let

$$v_\epsilon(x) = \begin{cases} \eta A w_\epsilon \frac{x - x_0}{\eta} + A x_0, & x \in \partial \eta(x_0), \\ A x, & x \in \Omega \setminus \partial \eta(x_0). \end{cases}$$

Clearly $v_\epsilon(x) = A x$ on $\partial \Omega \cup \partial \eta(x_0)$. Since $w_\epsilon \in W^{1,p}(\partial \eta(x_0))$ for $p \in [1, n)$, it follows that $v_\epsilon \in W^{1,p}(\Omega)$ for $p \in [1, n)$. Hence in particular, if $z_\epsilon = v_\epsilon|_{\partial \eta}$, then $z_\epsilon \in \mathcal{E}^\epsilon_{\epsilon_j}$ for $\epsilon > 0$. An easy calculation now shows that

$$\det \nabla z_\epsilon = \begin{cases} d_\epsilon \det A, & x \in \partial \eta(x_0) \setminus \partial \eta(x_0), \\ \det A, & x \in \Omega \setminus \partial \eta(x_0). \end{cases}$$

Using the definition of $d_\epsilon$, we get now that

$$\int_{\partial \Omega} \det \nabla z_\epsilon \, dx = (\det A) |\Omega| - V,$$

that is, $c_\epsilon(z_\epsilon) = 0$. Hence $z_\epsilon \in \mathcal{E}^\epsilon_{\epsilon_j}$.

For the second part of the lemma, we observe that for any nonnegative sequence $(\epsilon_j)$ converging to zero, we can conclude from [1, Eqn. (4.5)] that $(v_{\epsilon_j})$ is a Cauchy sequence in $W^{1,p}(\Omega)$. Since $v_{\epsilon_j} \to v_0$ a.e., we have that $v_{\epsilon_j} \to v_0$ in $W^{1,p}(\Omega)$. From (60) and since $(\det \nabla v_{\epsilon_j})$ converges a.e. to

$$\begin{cases} d_0 \det A, & x \in \partial \eta(x_0), \\ \det A, & x \in \Omega \setminus \partial \eta(x_0). \end{cases}$$

we get that

$$E_{\epsilon_j}(z_{\epsilon_j}) = \int_{\partial \Omega} W(\nabla z_{\epsilon_j}) \, dx \leq \int_{\Omega} W(\nabla v_{\epsilon_j}) \, dx \leq K \left( \left\| \nabla v_{\epsilon_j} \right\| + 1 \right) \leq C,$$

for some constant $C > 0$. \qed

References

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