On the Global Stability of Two-dimensional, Incompressible, Elastic Bars in Uniaxial Extension

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When a rectangular bar is subjected to uniaxial tension, the bar usually deforms (approximately) homogeneously and isoaxially until a critical load is reached. A bifurcation, such as the formation of shear bands or a neck, may then be observed. In this paper such an experiment is modelled as the in-plane extension of a two-dimensional, homogeneous, isotropic, incompressible, hyperelastic material in which the length of the bar is prescribed, the ends of the bar are assumed to be free of shear, and the sides are left completely free. It is shown that standard, additional constitutive hypotheses on the stored-energy function imply that no such bifurcation is possible in this model due to the fact that the homogeneous isoaxial deformation is the unique absolute minimiser of the elastic energy. Thus, in order for a bifurcation to occur either the material must cease to be elastic or the stored-energy function must violate the additional hypotheses. The fact that no local bifurcations can occur under the assumptions used herein was known previously, since these assumptions prohibit the load on the bar from reaching a maximum value. However, the fact that the homogeneous deformation is the absolute minimiser of the energy appears to be a new result.

Keywords: Incompressible, elastic, uniaxial tension, homogeneous absolute minimiser.

1. Introduction

Consider a homogeneous, isotropic, incompressible, hyperelastic material that occupies the rectangular region

$$\mathscr{R} := \{(x,y) : -R < x < R,\ 0 < y < L\}$$

in a fixed homogeneous reference configuration. A deformation $\mathbf{u} : \overline{\mathscr{R}} \to \mathbb{R}^2$ of the body is then a differentiable, one-to-one map that satisfies the constraint

$$\det \nabla\mathbf{u} \equiv 1 \quad \text{on } \overline{\mathscr{R}}. \tag{1.1}$$

The problem we herein consider is uniaxial extension. Specifically, we fix $\lambda \geq 1$ and restrict our attention to those deformations that satisfy the boundary conditions:

$$u_2(x,0) = 0, \quad u_2(x,L) = \lambda L \quad \text{for all } x \in [-R,R], \tag{1.2}$$

where we have written

$$\mathbf{u}(x,y) = \begin{bmatrix} u_1(x,y) \\ u_2(x,y) \end{bmatrix}.$$

With each such deformation we associate a corresponding elastic energy

$$E(\mathbf{u}) = \int_{\mathscr{R}} W(\nabla\mathbf{u}(x,y))\,dx\,dy, \tag{1.3}$$

where $\nabla\mathbf{u}$ denotes the 2×2 matrix of partial derivatives of $\mathbf{u}$, $W : \mathrm{M}_1^{2\times 2} \to [0,\infty)$ is the stored-energy density, and $\mathrm{M}_1^{2\times 2}$ denotes the set of 2×2 matrices with determinant equal to 1. If W is both isotropic and frame-indifferent, then standard representation theorems (see, e.g., Ciarlet 1988 or Gurtin 1981) imply that there is a function $\Phi : \mathbb{R}^+ \to \mathbb{R}$ that satisfies

$$W(\mathbf{F}) = \Phi(|\mathbf{F}|) \ \text{ for all } \mathbf{F} \in \mathrm{M}_1^{2\times 2}, \tag{1.4}$$

where $|\mathbf{F}|$ denotes the square-root of the sum of the squares of the elements of $\mathbf{F}$. Note that if a deformation $\mathbf{u}$ satisfies (1.1), (1.2) and minimises (1.3), (1.4), then so does $\mathbf{g}\circ\mathbf{u}$ where $\mathbf{g}$ is any translation in the x-direction. In order to eliminate this trivial nonuniqueness we impose the additional constraint

$$\int_{\mathscr{R}} u_1(x,y)\,dx\,dy = 0. \tag{1.5}$$

Our main result shows that if the function Φ is both convex and monotone increasing†, then the homogeneous deformation

$$\mathbf{u}^{\mathrm{h}}_{\lambda}(x,y) := \begin{bmatrix} \frac{1}{\lambda}x \\ \lambda y \end{bmatrix} \tag{1.6}$$

is an absolute minimiser of E. Moreover, if in addition Φ is strictly increasing, then $\mathbf{u}^{\mathrm{h}}_{\lambda}$ is the only absolute minimiser of the elastic energy that satisfies (1.1), (1.2), and (1.5). Furthermore, in this case (1.6) is therefore a strict local minimiser of the energy and so no bifurcations can occur.

The proof of our result uses the technique developed in Sivaloganathan & Spector (2009*a*) for energy minimisation of thick spherical shells. The main idea is fairly simple. We first consider the stored-energy function $W(\mathbf{F}) = |\mathbf{F}|$. For this function we show that the constraint of incompressibility allows us to bound the elastic energy below by an integral of a convex function of the deformed length of line segments that were initially parallel to the loading axis. Moreover, this lower bound is an equality when the image curves are straight lines that are deformed uniformly and are parallel to the loading axis. Thus, energetically, the material *prefers* that each such straight line deform homogeneously into another parallel straight line. The general case then follows from Jensen's inequality applied to the convex, increasing function Φ.

We note that the final remark in this paper demonstrates that our constitutive assumptions include the Ogden (1972) materials

$$W(\mathbf{F}) = \sum_{k=1}^{M} \frac{\mu_k}{\alpha_k}\left[\lambda_1^{\alpha_k} + \lambda_2^{\alpha_k}\right],$$

provided $\mu_k > 0$ and $\alpha_k \geq 2$. Here λ_1 and λ_2 are the principal stretches, i.e., the eigenvalues of $\mathbf{V} = \sqrt{\mathbf{F}\mathbf{F}^{\mathrm{T}}}$. One of the simplest such examples is the neo-Hookean material:

$$W(\mathbf{F}) = \tfrac{\mu}{2}|\mathbf{F}|^2 = \tfrac{\mu}{2}\left[\lambda_1^2 + \lambda_2^2\right],$$

† If Φ is convex and monotone increasing, then W is polyconvex in the sense of Ball (1977).

which clearly satisfies our hypotheses.

The vast majority of prior results on elastic bars in uniaxial tension have analysed the linearization stability of $\mathbf{u}^{\mathrm{h}}_{\lambda}$, that is, whether or not the system of partial differential equations that one obtains upon linearising the equilibrium equations, (1.2), and (1.5), about $\mathbf{u}^{\mathrm{h}}_{\lambda}$, has a nontrivial solution. This technique was utilised by Wesołowski (1962) to show that a neo-Hookean material is always stable in tension, while certain other constitutive relations do become unstable. Hill and Hutchinson (1975) then employed this procedure to prove that an incompressible elastic material is linearization stable in tension as long as the linear-elasticity tensor remains elliptic and the force required to extend the rectangle is an increasing function of the extension ratio λ. For compressible elastic materials similar results have been obtained by Del Piero (1980) and also in Spector (1984). Recent results of Del Piero and Rizzoni (2008) examine the stability of both compressible and incompressible materials. There is also an extensive literature on plastic and elastic/plastic materials, see, e.g., Guduru & Freund (2002), Hill & Hutchinson (1975), and the references therein.

We mention that linearization instability does not necessitate bifurcation. The additional technical details needed to establish the existence of a second solution branch can be found in Simpson & Spector (2008). Alternatively, another approach to this problem was proposed by Mielke (1991) who used the centre-manifold theorem to prove that an infinite strip will bifurcate when the force required to extend the strip has a local maximum as a function of λ. When a bar is compressed rather than extended, $\lambda \in (0,1)$, many authors have investigated this and many similar problems; see, e.g., Davies (1989) and the references therein.

Finally, we note that it is unclear whether or not one can use techniques similar to those in Sivaloganathan & Spector (2009*a*, *b*) and this paper to extend our results to compressible materials and also to three-dimensional circular-cylindrical bars and shells.

2. Isochoric Deformations

Definition 2.1. We let $\lambda > 0$ and define the set of *admissible isochoric deformations* by

$$\mathscr{A}_\lambda := \left\{ \mathbf{u} \in C^1(\overline{\mathscr{R}};\mathbb{R}^2) : \det \nabla\mathbf{u} \equiv 1,\ \mathbf{u} \text{ is one-to-one, } \ \mathbf{u} \text{ satisfies (1.2) and (1.5)} \right\}.$$

In particular, the *homogeneous* deformation (1.6) is an admissible deformation that satisfies

$$\nabla\mathbf{u}^{\mathrm{h}}_{\lambda} \equiv \begin{bmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{bmatrix}, \qquad |\nabla\mathbf{u}^{\mathrm{h}}_{\lambda}|^2 \equiv \lambda^2 + \lambda^{-2}. \tag{2.1}$$

The shortest curve connecting two parallel lines is any perpendicular straight line segment that connects the lines. We will need a slight variant of this well-known result. We provide a proof for the convenience of the reader.

Lemma 2.2. *Let $\lambda > 0$, $\mathbf{u} \in \mathscr{A}_\lambda$, and suppose that $\mathbf{u}^{\mathrm{h}}_{\lambda}$ is given by* (1.6)*. Then, for each $x \in [-R,R]$,*

$$\text{length}\Big(\mathbf{u}(\mathscr{L}_x)\Big) \geq \text{length}\Big(\mathbf{u}^{\mathrm{h}}_{\lambda}(\mathscr{L}_x)\Big),$$

where $\mathscr{L}_x$ is the line segment

$$\mathscr{L}_x := \{(x,y) : 0 \leq y \leq L\}.$$

Moreover, if for every $x \in [-R,R]$ this inequality is an equality then $\mathbf{u} \equiv \mathbf{u}^{\mathrm{h}}_{\lambda}$.

Proof. Let $\lambda > 0$, $\mathbf{u} \in \mathscr{A}_\lambda$, and $x \in [-R,R]$. Then

$$\text{length}\Big(\mathbf{u}(\mathscr{L}_x)\Big) = \int_0^L \left|\frac{\partial \mathbf{u}}{\partial y}\right| dy \geq \int_0^L \left|\frac{\partial u_2}{\partial y}\right| dy \geq \left|\int_0^L \frac{\partial u_2}{\partial y}\, dy\right|. \tag{2.2}$$

However,

$$\left|\int_0^L \frac{\partial u_2}{\partial y}\, dy\right| = |u_2(x,L) - u_2(x,0)| = \lambda L = \text{length}\Big(\mathbf{u}_\lambda^{\mathrm{h}}(\mathscr{L}_x)\Big),$$

where we have made use of $u_2(\cdot,L) \equiv \lambda L$, $u_2(\cdot,L) \equiv 0$, and (1.6).

To prove strictness when $\mathbf{u} \not\equiv \mathbf{u}_\lambda^{\mathrm{h}}$, we note that the first inequality in (2.2) is a strict inequality unless $\partial u_1/\partial y \equiv 0$. Consequently, $u_1(x,y) = \rho(x)$ for some function $\rho : [-R,R] \to \mathbb{R}$. However, since $\mathbf{u}$ is isochoric it follows that $u_2(x,y) = \widehat{\rho}(x) + y/\rho'(x)$ for some function $\widehat{\rho} : [-R,R] \to \mathbb{R}$. We next apply the boundary condition $u_2(\cdot,0) \equiv 0$ to conclude that $\widehat{\rho} = 0$. The boundary condition $u_2(\cdot,L) \equiv \lambda L$ then yields $\rho' \equiv 1/\lambda$. Therefore

$$\mathbf{u}(x,y) = \begin{bmatrix} \frac{1}{\lambda}x + a \\ \lambda y \end{bmatrix}$$

for some $a \in \mathbb{R}$. Finally, (1.5) implies that $a = 0$ and so $\mathbf{u} \equiv \mathbf{u}_\lambda^{\mathrm{h}}$. $\square$

3. The Homogeneity of Isochoric Energy-Minimising Deformations

Let $\mathbf{u} \in \mathscr{A}_\lambda$. Our aim is to prove that the energy functional (1.3) satisfies

$$E(\mathbf{u}) \geq E(\mathbf{u}_\lambda^{\mathrm{h}})$$

for any polyconvex stored-energy function W of the form

$$W(\mathbf{F}) = \Phi(|\mathbf{F}|), \tag{3.1}$$

where $\Phi : \mathbb{R}^+ \to \mathbb{R}$ is convex and monotone increasing.

To present the main ideas in our proof we present our results first for the energy $W(\mathbf{F}) = |\mathbf{F}|$. The more general result will then be a consequence of Jensen's inequality. To simplify the technical details we will assume that $\mathbf{u} \in C^1(\overline{\mathscr{R}};\mathbb{R}^2)$, however, the proofs easily generalise to wider classes of deformations in an appropriate Sobolev space.

(*a*) *The case* $W(\mathbf{F}) = |\mathbf{F}|$.

Lemma 3.1. *Let $\lambda > 0$ and $\mathbf{u} \in \mathscr{A}_\lambda$. Then, for $(x,y) \in \overline{\mathscr{R}}$,*

$$|\nabla \mathbf{u}|^2 \geq \frac{1}{\left|\frac{\partial \mathbf{u}}{\partial y}\right|^2} + \left|\frac{\partial \mathbf{u}}{\partial y}\right|^2. \tag{3.2}$$

Proof. We first observe that

$$|\nabla \mathbf{u}(x,y)|^2 = \left|\frac{\partial \mathbf{u}}{\partial x}\right|^2 + \left|\frac{\partial \mathbf{u}}{\partial y}\right|^2. \tag{3.3}$$

Next, since $\mathbf{u}$ is isochoric the Cauchy-Schwarz inequality implies

$$1 = \det \nabla \mathbf{u} = \frac{\partial \mathbf{u}}{\partial y} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\partial \mathbf{u}}{\partial x} \leq \left| \frac{\partial \mathbf{u}}{\partial y} \right| \left| \frac{\partial \mathbf{u}}{\partial x} \right| \tag{3.4}$$

since the above matrix is orthogonal. The desired result now follows from (3.3) and (3.4). □

A straightforward computation then gives us the following result.

Lemma 3.2. *Let*

$$g(t) = \sqrt{\frac{1}{t^2} + t^2} \ \text{for} \ \ t \in (0, \infty).$$

Then g is convex on $(0,\infty)$ and strictly monotone increasing for $t \geq 1$.

Lemma 3.3. *Let $\lambda > 0$ and $\mathbf{u} \in \mathscr{A}_\lambda$. Then, for each $x \in [-R,R]$,*

$$⨍_0^L |\nabla \mathbf{u}|\, dy \geq ⨍_0^L g\left(\left| \frac{\partial \mathbf{u}}{\partial y} \right| \right) dy \geq g\left(⨍_0^L \left| \frac{\partial \mathbf{u}}{\partial y} \right| dy \right), \tag{3.5}$$

where $⨍_0^L \phi \, dy$ denotes the average value of ϕ over $[0,L]$, i.e.,

$$⨍_0^L \phi(x,y)\, dy := \frac{1}{L} \int_0^L \phi(x,y)\, dy.$$

Proof. If we take the square-root of (3.2) and then integrate the result over $[0,L]$ and divide by L, the result will follow from Jensen's inequality since g is convex by Lemma 3.2. □

Lemma 3.4. *Let $\lambda > 0$ and $\mathbf{u} \in \mathscr{A}_\lambda$. Then, for each $x \in [-R,R]$,*

$$⨍_0^L \left| \frac{\partial \mathbf{u}}{\partial y} \right| dy \geq ⨍_0^L \left| \frac{\partial \mathbf{u}_\lambda^{\mathrm{h}}}{\partial y} \right| dy = \lambda.$$

Moreover, if for every $x \in [-R,R]$ this inequality is an equality then $\mathbf{u} \equiv \mathbf{u}_\lambda^{\mathrm{h}}$.

Proof. This result is an immediate consequence of Lemma 2.2 since

$$\int_0^L \left| \frac{\partial \mathbf{v}}{\partial y} \right| dy = \mathrm{length}\Big(\mathbf{v}\big(x \times [0,L]\big) \Big)$$

for any $\mathbf{v} \in \mathscr{A}_\lambda$ and $x \in [-R,R]$. □

If we now combine Lemmas 3.2–3.4 and make use of the identity (see $(2.1)_2$)

$$g\left(⨍_0^L \left| \frac{\partial \mathbf{u}_\lambda^{\mathrm{h}}}{\partial y} \right| dy \right) = g(\lambda) \equiv |\nabla \mathbf{u}_\lambda^{\mathrm{h}}(x,y)|,$$

we arrive at the following result.

Proposition 3.5. *Let $\lambda \geq 1$, $\mathbf{u} \in \mathscr{A}_\lambda$, and $x \in [-R,R]$. Then*

$$⨍_0^L |\nabla \mathbf{u}(x,y)|\, dy \geq ⨍_0^L |\nabla \mathbf{u}_\lambda^{\mathrm{h}}(x,y)|\, dy. \tag{3.6}$$

Moreover, if for every $x \in [-R,R]$ this inequality is an equality then $\mathbf{u} \equiv \mathbf{u}_\lambda^{\mathrm{h}}$.

(b) The general case: $W(\mathbf{F}) = \Phi(|\mathbf{F}|)$.

We now suppose that $W(\mathbf{F}) = \Phi(|\mathbf{F}|)$ to obtain the following result.

Theorem 3.6. *Let* $\lambda \geq 1$ *and*

$$W(\mathbf{F}) = \Phi(|\mathbf{F}|),$$

where $\Phi : \mathbb{R}^+ \to \mathbb{R}$ *is convex and monotone increasing on* $[\sqrt{2}, \infty)$*. Then, for any* $\mathbf{u} \in \mathscr{A}_\lambda$*,*

$$E(\mathbf{u}) = \int_{\mathscr{R}} W(\nabla \mathbf{u})\, dx\, dy \geq \int_{\mathscr{R}} W(\nabla \mathbf{u}_\lambda^{\mathrm{h}})\, dx\, dy = E(\mathbf{u}_\lambda^{\mathrm{h}}).$$

Moreover, if in addition Φ *is strictly increasing, then the inequality is strict when* $\mathbf{u} \not\equiv \mathbf{u}_\lambda^{\mathrm{h}}$*.*

Proof. We first note that the constraint $\det \mathbf{F} = 1$ implies $|\mathbf{F}| \geq \sqrt{2}$. Therefore, $|\nabla \mathbf{u}(x,y)| \geq \sqrt{2}$ and $|\nabla \mathbf{u}_\lambda^{\mathrm{h}}(x,y)| \geq \sqrt{2}$ for every $(x,y) \in \mathscr{R}$. Next, by Jensen's inequality, the monotonicity of Φ, Proposition 3.5, and $(2.1)_2$

$$\begin{aligned}
\int_{\mathscr{R}} W(\nabla \mathbf{u})\, dx\, dy &= L \int_{-R}^{R} \left(⨍_0^L \Phi(|\nabla \mathbf{u}|)\, dy \right) dx \\
&\geq L \int_{-R}^{R} \Phi \left(⨍_0^L |\nabla \mathbf{u}|\, dy \right) dx \\
&\geq L \int_{-R}^{R} \Phi \left(⨍_0^L |\nabla \mathbf{u}_\lambda^{\mathrm{h}}|\, dy \right) dx \\
&= L \int_{-R}^{R} \Phi \left(|\nabla \mathbf{u}_\lambda^{\mathrm{h}}| \right) dx = \int_{\mathscr{R}} W(\nabla \mathbf{u}_\lambda^{\mathrm{h}})\, dx\, dy.
\end{aligned}$$

In order to see that the inequality is strict when $\mathbf{u} \not\equiv \mathbf{u}_\lambda^{\mathrm{h}}$ we observe that Proposition 3.5 and the strict monotonicity of Φ imply that the second of the above inequalities is a strict inequality when $\mathbf{u} \not\equiv \mathbf{u}_\lambda^{\mathrm{h}}$. $\square$

Remark 3.7. Under appropriate growth conditions on W (see Ball 1977; Ciarlet & Nečas 1987; Ciarlet 1988), $\mathbf{u}_\lambda^{\mathrm{h}}$ is also the (strict) minimiser of the energy E over the larger sets:

$$\{\mathbf{u} \in W^{1,p}(\mathscr{R};\mathbb{R}^2) : \det \nabla \mathbf{u} = 1 \ a.e.,\ \mathbf{u} \text{ is one-to-one } a.e.,\ \mathbf{u} \text{ satisfies (1.2) and (1.5)}\},$$

$p \in [1, \infty]$. The details are similar to those used in §5 of Sivaloganathan & Spector (2009*a*).

Remark 3.8. The (Piola-Kirchhoff) stress for the constitutive model used in this paper is given by

$$\mathbf{S}(\mathbf{F}) := \frac{dW}{d\mathbf{F}} - \pi [\operatorname{adj} \mathbf{F}]^{\mathrm{T}} = \Phi'(|\mathbf{F}|) \frac{\mathbf{F}}{|\mathbf{F}|} - \pi [\operatorname{adj} \mathbf{F}]^{\mathrm{T}} \ \text{ for all } \mathbf{F} \in \mathrm{M}_1^{2\times 2},$$

where $\pi \in \mathbb{R}$ is a pressure. In particular when $\mathbf{F} = \nabla \mathbf{u}_\lambda^{\mathrm{h}}$ it follows that the force (per unit undeformed length) on the sides of the rectangle is

$$S_{11} = \frac{1}{\lambda} \frac{\Phi'(\sqrt{\lambda^{-2} + \lambda^2})}{\sqrt{\lambda^{-2} + \lambda^2}} - \lambda \pi = 0,$$

while the force (per unit undeformed length) on the top and bottom is

$$S_{22} = \lambda \frac{\Phi'(\sqrt{\lambda^{-2} + \lambda^2})}{\sqrt{\lambda^{-2} + \lambda^2}} - \frac{1}{\lambda} \pi.$$

If we combine these two equations we conclude that

$$S_{22}(\lambda) = \Phi'\left(\sqrt{\lambda^{-2}+\lambda^2}\right)\frac{\lambda-\lambda^{-3}}{\sqrt{\lambda^{-2}+\lambda^2}}$$

and hence

$$\frac{dS_{22}}{d\lambda} = \frac{2\lambda^{-6}+6\lambda^{-2}}{\left(\lambda^2+\lambda^{-2}\right)^{3/2}}\Phi'\left(\sqrt{\lambda^{-2}+\lambda^2}\right) + \frac{\left(\lambda-\lambda^{-3}\right)^2}{\lambda^{-2}+\lambda^2}\Phi''\left(\sqrt{\lambda^{-2}+\lambda^2}\right),$$

which is nonnegative if Φ is convex and monotone increasing. Thus the force increases with increasing extension, which has been shown by Hill and Hutchinson (1975) (see, also, Del Piero 1980; Del Piero & Rizzoni 2008; Spector 1984) to be a sufficient condition to prohibit bifurcation of an incompressible elastic material that has an elasticity tensor that is elliptic.

Remark 3.9. Among the many papers in the literature that develop constitutive relations for rubber-like materials, one of the most influential is that of Ogden (1972). The energies developed there are of the form:

$$W(\mathbf{F}) = \phi(\lambda_1)+\phi(\lambda_2),$$

where λ_i are the principal stretches, i.e., the eigenvalues of $\mathbf{V}=\sqrt{\mathbf{F}\mathbf{F}^{\mathrm{T}}}$ and $\phi : (0,\infty)\to\mathbb{R}$. If ϕ is convex and monotone increasing, then such energies are polyconvex (see pp. 363-367 in Ball 1977). In order to compare restrictions on ϕ and Φ we note that the eigenvalues of $\mathbf{V}$ satisfy the characteristic equation (recall $\det\mathbf{V}=1$)

$$\lambda^2 - \mathrm{tr}(\mathbf{V})\lambda + 1 = 0,$$

where $\mathrm{tr}(\mathbf{V}) = \lambda_1+\lambda_2$ denotes the trace of $\mathbf{V}$. The identity $[\mathrm{tr}(\mathbf{V})]^2 = |\mathbf{F}|^2+2$ together with the quadratic formula then yield

$$\lambda_{\mathrm{L}}(t) = \tfrac{1}{2}\left[\sqrt{t^2+2}+\sqrt{t^2-2}\right], \qquad \lambda_{\mathrm{S}}(t) = \tfrac{1}{2}\left[\sqrt{t^2+2}-\sqrt{t^2-2}\right] \tag{3.7}$$

for t:=$|\mathbf{F}|\geq\sqrt{2}$. Consequently, if we differentiate (3.7) with respect to t we find that

$$\dot\lambda_{\mathrm{L}}(t) = \frac{t}{\sqrt{t^4-4}}\lambda_{\mathrm{L}}(t), \qquad \dot\lambda_{\mathrm{S}}(t) = -\frac{t}{\sqrt{t^4-4}}\lambda_{\mathrm{S}}(t), \tag{3.8}$$

and hence

$$\ddot\lambda_{\mathrm{L}}(t) = \left[\frac{t^2}{t^4-4} - \frac{t^4+4}{(t^4-4)^{3/2}}\right]\lambda_{\mathrm{L}}(t), \qquad \ddot\lambda_{\mathrm{S}}(t) = \left[\frac{t^2}{t^4-4} + \frac{t^4+4}{(t^4-4)^{3/2}}\right]\lambda_{\mathrm{S}}(t). \tag{3.9}$$

Thus $t\mapsto\lambda_{\mathrm{L}}(t)$ is monotone increasing and concave while $t\mapsto\lambda_{\mathrm{S}}(t)$ is monotone decreasing and convex.

We observe that our constitutive relation can be written

$$W(\mathbf{F}) = \Phi(|\mathbf{F}|), \qquad \Phi(t) := \phi\big(\lambda_{\mathrm{L}}(t)\big)+\phi\big(\lambda_{\mathrm{S}}(t)\big). \tag{3.10}$$

Differentiating the last equation with respect to t yields

$$\dot\Phi(t) = \phi'\big(\lambda_{\mathrm{L}}(t)\big)\dot\lambda_{\mathrm{L}}(t)+\phi'\big(\lambda_{\mathrm{S}}(t)\big)\dot\lambda_{\mathrm{S}}(t), \tag{3.11}$$

which together with (3.7) and (3.8) implies that

$$\dot{\Phi}(t) = \frac{t}{2\sqrt{t^2+2}}\Big[\phi'(\lambda_{\rm L}) + \phi'(\lambda_{\rm S})\Big] + \frac{t}{2\sqrt{t^2-2}}\Big[\phi'(\lambda_{\rm L}) - \phi'(\lambda_{\rm S})\Big]. \tag{3.12}$$

Thus if ϕ is convex and increasing, then Φ will be monotone increasing on $[\sqrt{2},\infty)$. In particular this is true when (cf. pp. 570–571 in Ogden 1972)

$$\phi(\lambda) := \sum_{i=1}^{M} \frac{\mu_i}{\alpha_i}\lambda^{\alpha_i}, \tag{3.13}$$

with $\mu_i > 0$ and $\alpha_i \geq 1$. Moreover, if we differentiate (3.11) with respect to t we arrive at

$$\ddot{\Phi}(t) = \phi''(\lambda_{\rm L})\big[\dot{\lambda}_{\rm L}\big]^2 + \phi''(\lambda_{\rm S})\big[\dot{\lambda}_{\rm S}\big]^2 + \phi'(\lambda_{\rm L})\ddot{\lambda}_{\rm L} + \phi'(\lambda_{\rm S})\ddot{\lambda}_{\rm S}.$$

Thus if ϕ is given by (3.13) it follows from (3.8) and (3.9) that $\ddot{\Phi}(t)$ will be the sum of terms of the form

$$\frac{N(t)}{\big(t^4-4\big)^{3/2}}, \tag{3.14}$$

$$N(t) := \alpha t^2\sqrt{t^4-4}\Big(\big[\lambda_{\rm L}(t)\big]^\alpha + \big[\lambda_{\rm S}(t)\big]^\alpha\Big) + \big(t^4+4\big)\Big(\big[\lambda_{\rm S}(t)\big]^\alpha - \big[\lambda_{\rm L}(t)\big]^\alpha\Big).$$

We observe that the denominator of (3.14) is strictly positive on $(\sqrt{2},\infty)$, $N(\sqrt{2}) = 0$, and

$$\frac{dN}{dt} = 3\alpha t\sqrt{t^4-4}\Big(\big[\lambda_{\rm L}(t)\big]^\alpha + \big[\lambda_{\rm S}(t)\big]^\alpha\Big) + t^3\big(\alpha^2-4\big)\Big(\big[\lambda_{\rm L}(t)\big]^\alpha - \big[\lambda_{\rm S}(t)\big]^\alpha\Big), \tag{3.15}$$

which is strictly positive on $(\sqrt{2},\infty)$ when $\alpha \geq 2$. The numerator of (3.14) is therefore also strictly positive on $(\sqrt{2},\infty)$ for $\alpha \geq 2$.

We therefore conclude that *the constitutive relation given by* (3.7)*,* (3.10)*, and* (3.13)*, with $\mu_i > 0$ and $\alpha_i \geq 2$, satisfies* $(3.10)_1$*, where Φ is convex and strictly monotone increasing on $[\sqrt{2},\infty)$. It therefore follows from Theorem 3.6 that $\mathbf{u}^{\rm h}_\lambda$, given by* (1.6)*, is the unique absolute minimiser of these energies for every $\lambda \geq 1$.*

Remark 3.10. Finally, we note that in order for W to be polyconvex the function Φ must be convex and monotone increasing on $[0,\infty)$. In particular, if one is given $\Psi = \Psi(\lambda_1,\lambda_2)$, expressed as a function of the principal stretches, different extensions of Ψ off the curve $\lambda_1\lambda_2 = 1$ will in general yield different representations for $\Phi(|\mathbf{F}|) = \Psi(\lambda_1,\lambda_2)$. The resulting Φ may or may not be convex and increasing on all of $[0,\infty)$ depending on the extension. For example, although the previous remark demonstrates that the Ogden material $\Psi(\lambda_1,\lambda_2) = \lambda_1^6 + \lambda_2^6$ does satisfy our conditions on $[\sqrt{2},\infty)$, a direct computation using $\lambda_1\lambda_2 = 1$ shows that

$$\Psi(\lambda_1,\lambda_2) := \lambda_1^6 + \lambda_2^6 = \big(\lambda_1^2+\lambda_2^2\big)^3 - 3\big(\lambda_1^2+\lambda_2^2\big) = |\mathbf{F}|^6 - 3|\mathbf{F}|^2 =: \Phi(|\mathbf{F}|),$$

which is neither convex nor monotone increasing all of $[0,\infty)$. However, the constraint $\det\mathbf{F} = 1$ implies that $|\mathbf{F}| \geq \sqrt{2}$ and so all one need show is that Φ restricted to $[\sqrt{2},\infty)$ has an extension to $[0,\infty)$ that is convex and increasing, which is straightforward for this Ψ.

To see that such an extension exists when ϕ is given by (3.13) we observe that (3.7), (3.8), (3.12), and (3.13) yield, with the aid of l'Hôpital's rule,

$$\lim_{t \searrow \sqrt{2}} \dot{\Phi}(t) = \frac{\sqrt{2}}{2}\left[\phi'(1) + \phi''(1)\right] = \frac{\sqrt{2}}{2}\sum_{i=1}^{M} \mu_i \alpha_i > 0$$

for $\mu_i \alpha_i > 0$. Similarly, (3.7), (3.8), (3.13), (3.14), (3.15) together with two applications of l'Hôpital's rule yield, for $\mu_i \alpha_i > 0$ and $\alpha_i \geq 2$,

$$\lim_{t \searrow \sqrt{2}} \ddot{\Phi}(t) = \sum_{i=1}^{M}\left[\mu_i \alpha_i \left(\frac{1}{2} + \frac{\alpha_i^2 - 4}{12}\right)\right] > 0.$$

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