On the Symmetry of Energy Minimising Deformations in Nonlinear Elasticity I: Incompressible Materials

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ABSTRACT. Consider an incompressible, nonlinear, hyperelastic material which occupies the region $A \subset \mathbb{R}^n$, $n \geq 2$, in its reference configuration, where A denotes the annular region

$$A = \{\mathbf{x} \in \mathbb{R}^n : a < |\mathbf{x}| < b\},$$

$0 < a < b$. Deformations of A are therefore isochoric maps $\mathbf{u} : A \to \mathbb{R}^n$ and so satisfy the incompressibility constraint

$$\det \nabla \mathbf{u} = 1.$$

The boundary of the annulus ∂A is separated into two disjoint pieces $\partial A = \partial A_o \cup \partial A_I$, where $\partial A_I = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = a\}$ and $\partial A_o = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = b\}$ denote the inner and outer boundary components respectively. We study displacement and mixed displacement/zero-traction boundary-value problems in which we impose a displacement boundary condition of the form

$$\mathbf{u}(\mathbf{x}) = \sigma \mathbf{x}$$

on one of the boundary components (where $\sigma > 0$ is a given constant) and the displacement on the remaining boundary component is either prescribed (in the case of the pure displacement boundary-value problem) or left unspecified (in the case of the mixed boundary-value problem).

In this paper we use isoperimetric arguments to prove that the radially symmetric solutions to these problems are global energy minimisers in various classes of (possibly non-symmetric) isochoric deformations of the annulus.

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1. Introduction

Consider an incompressible, nonlinearly elastic body that occupies the annular region

$$A = \{\mathbf{x} \in \mathbb{R}^n : a < |\mathbf{x}| < b\ \}, \tag{1.1}$$

$0 < a < b$, in its reference configuration (the physically relevant values of n are 2 and 3). An admissible deformation of the body corresponds to a mapping $\mathbf{u} : A \to \mathbb{R}^n$ which is one-to-one almost everywhere and satisfies the incompressibility[1] constraint

$$\det \nabla\mathbf{u} = 1 \quad \text{for a.e. } \mathbf{x} \in A. \tag{1.2}$$

Deformations satisfying the above condition are known as *isochoric* deformations. In nonlinear hyperelasticity, with each such deformation we associate a corresponding energy

$$E(\mathbf{u}) = \int_A W(\nabla\mathbf{u}(\mathbf{x}))\, d\mathbf{x}, \tag{1.3}$$

where $W : \mathrm{M}_1^{n\times n} \to [0,\infty)$ is the stored-energy function and $\mathrm{M}_1^{n\times n}$ denotes the set of $n \times n$ matrices with determinant equal to one. If W is both isotropic and frame indifferent then

$$W(\mathbf{FQ}) = W(\mathbf{QF}) = W(\mathbf{F}) \quad \text{for all } \mathbf{F} \in \mathrm{M}_1^{n\times n} \text{ and } \mathbf{Q} \in SO(n),$$

where $SO(n)$ denotes the special orthogonal group of $n \times n$ matrices. We next recall the notion of a polyconvex stored-energy function in two and three dimensions:

- If $n = 2$, then W is polyconvex if

  $$W(\mathbf{F}) = G(\mathbf{F}) \quad \text{for all } \mathbf{F} \in \mathrm{M}_1^{2\times 2},$$

  where $G : M_+^{2\times 2} \to \mathbb{R}$ is convex;

- If $n = 3$, then W is polyconvex if

  $$W(\mathbf{F}) = G(\mathbf{F}, \operatorname{adj}\mathbf{F}) \quad \text{for all } \mathbf{F} \in \mathrm{M}_1^{3\times 3},$$

  where $G : \mathrm{M}_+^{3\times 3} \times \mathrm{M}_+^{3\times 3} \to \mathbb{R}$ is convex and $\operatorname{adj}\mathbf{F}$ denotes the adjugate matrix[2] of $\mathbf{F}$.

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[1] Vulcanized rubber is often modelled as an incompressible material.

[2] That is, the unique 3×3 matrix satisfying $(\operatorname{adj}\mathbf{F})\mathbf{F} = \mathbf{I}$ for each $\mathbf{F} \in M_1^{3\times 3}$.

The results contained in this paper apply to stored-energy functions that have the following forms:

- If $n = 2$, then
$$W(\mathbf{F}) = \Phi(|\mathbf{F}|), \tag{1.4}$$
where $\Phi : \mathbb{R}^+ \to \mathbb{R}$ is convex and monotone increasing;

- If $n = 3$, then
$$W(\mathbf{F}) = \Phi(|\mathbf{F}|^2,\ |\mathrm{adj}\,\mathbf{F}|), \tag{1.5}$$
where $\Phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is monotone increasing in each of its arguments and convex.[3]

Boundary Value Problems.

Let
$$\partial A_I = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = a\}, \qquad \partial A_o = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = b\}$$
denote the inner and outer boundaries of the annulus, respectively. We seek equilibrium states by minimising (1.3) on classes of admissible deformations satisfying given boundary conditions (pure displacement and mixed displacement/traction).

Pure Displacement Boundary Value Problem.

In the pure displacement boundary-value problem we take $\mu > 0$ and specify the conditions
$$\mathbf{u}(\mathbf{x}) = \mu\mathbf{x} \ \text{ for all } \mathbf{x} \in \partial A_I, \qquad \mathbf{u}(\mathbf{x}) = \lambda\mathbf{x} \ \text{ for all } \mathbf{x} \in \partial A_o, \tag{1.6}$$
where $\lambda > 0$ satisfies $\lambda^n b^n - \mu^n a^n = b^n - a^n$.

Mixed Displacement/Traction Boundary Value Problem.

In the mixed displacement/zero-traction boundary value problem we only consider tensile boundary conditions; we specify the condition
$$\mathbf{u}(\mathbf{x}) = \mu\mathbf{x} \ \text{ for all } \mathbf{x} \in \partial A_I, \tag{1.7}$$
where $\mu \geq 1$ and the outer boundary ∂A_o is left free; or the condition
$$\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x} \ \text{ for all } \mathbf{x} \in \partial A_o, \tag{1.8}$$
where $\lambda \geq 1$ and the inner boundary ∂A_I is left free.

[3] Note that all such stored-energy functions are frame-indifferent, isotropic, and polyconvex. This class of energy functions includes the neo-Hookean and Mooney-Rivlin energy functions.

For polyconvex stored-energy functions W, the existence theory of Ball [1] gives hypotheses under which a minimiser of (1.3) exists for either of the above problems in a general class of deformations satisfying (1.2).

In section 2 we show that the unique radial deformation satisfying (1.8) (or (1.7)) and (1.2) is the map

$$\mathbf{u}_\lambda^{\mathbf{rad}}(\mathbf{x}) = (R^n + b^n(\lambda^n - 1))^{\frac{1}{n}} \frac{\mathbf{x}}{|\mathbf{x}|} \quad \text{for all} \;\; \mathbf{x} \in \overline{A}. \tag{1.9}$$

In section 3 of this paper we use a symmetrisation argument based on isoperimetric estimates on deformed spheres to prove that the radial incompressible deformation (1.9) is a global minimiser of the energy (given by (1.3) and (1.5)) amongst all (possibly non-symmetric) C^1 isochoric deformations of the annulus. In section 5 we extend our arguments to deformations lying in the Sobolev space $W^{1,p}(A; \mathbb{R}^3)$, $p > 3$. We prove, in particular, that these radial deformations must coincide with the energy minimisers given by the existence theory of Ball in [1].

Finally, we note that the results in this paper yield a one-parameter family of inhomogeneous deformations, each of which is a global minimiser of a homogeneous energy for corresponding homogeneous boundary values. To our knowledge, the only other construction of inhomogeneous energy-minimising deformations is due to Zhang [28] who shows that, in a neighborhood of a linearization-stable, stress-free reference configuration, the solutions obtained by the implicit function theorem are indeed the global minimisers obtained by Ball [1].

For ease of exposition we will present the results in sections 3–5 for the case $n = 3$ and note the corresponding results and extensions for the case $n = 2$.

2. Radial Deformations of A.

It is well known (see, e.g., [4]) that if $\mathbf{u} \in C^1(\overline{A}; \mathbb{R}^n)$, is a radial deformation:

$$\mathbf{u}(\mathbf{x}) = \frac{r(R)}{R}\mathbf{x}, \qquad R := |\mathbf{x}|, \quad r : [a, b] \to [0, \infty),$$

then

$$\nabla\mathbf{u}(\mathbf{x}) = r'(R)\left(\frac{\mathbf{x} \otimes \mathbf{x}}{|\mathbf{x}|^2}\right) + \frac{r}{R}\left(\mathbf{I} - \frac{\mathbf{x} \otimes \mathbf{x}}{|\mathbf{x}|^2}\right). \tag{2.1}$$

Condition (1.2) then forces

$$r'(R)\left(\frac{r(R)}{R}\right)^{n-1} = 1, \tag{2.2}$$

from which it follows that $r(R) = (R^n + c^n)^{\frac{1}{n}}$, where c is a constant. Hence, the only kinematically admissible isochoric radial deformation satisfying (1.6) (or $(1.6)_1$ only) is

$$\mathbf{u}_\mu^{\mathbf{rad}}(\mathbf{x}) = \frac{r_\mu^{\mathrm{inc}}(R)}{R}\mathbf{x}, \tag{2.3}$$

$$r_\mu^{\mathrm{inc}}(R) := [R^n + a^n(\mu^n - 1)]^{\frac{1}{n}} = [R^n + b^n(\lambda^n - 1)]^{\frac{1}{n}} \tag{2.4}$$

(see, e.g., [4]). Dropping the subscript μ, for the moment, it then follows from (2.1) and (2.2) that

$$\nabla\mathbf{u}^{\mathbf{rad}}(\mathbf{x}) = \left(\frac{R}{r^{\mathrm{inc}}(R)}\right)^{n-1}\left(\frac{\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|^2}\right) + \frac{r^{\mathrm{inc}}(R)}{R}\left(\mathbf{I} - \frac{\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|^2}\right), \tag{2.5}$$

$$\mathrm{adj}\left(\nabla\mathbf{u}^{\mathbf{rad}}(\mathbf{x})\right) = \left(\frac{r^{\mathrm{inc}}(R)}{R}\right)^{n-1}\left(\frac{\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|^2}\right) + \frac{R}{r^{\mathrm{inc}}(R)}\left(\mathbf{I} - \frac{\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|^2}\right), \tag{2.6}$$

and hence that

$$|\nabla\mathbf{u}^{\mathbf{rad}}|^2 = \mathrm{tr}\left[(\nabla\mathbf{u}^{\mathbf{rad}})^{\mathrm{T}}\nabla\mathbf{u}^{\mathbf{rad}}\right] = \left(\frac{R}{r^{\mathrm{inc}}(R)}\right)^{2(n-1)} + (n-1)\left(\frac{r^{\mathrm{inc}}(R)}{R}\right)^2 \tag{2.7}$$

and

$$\left|\mathrm{adj}\left(\nabla\mathbf{u}^{\mathbf{rad}}\right)\right|^2 = \left(\frac{r^{\mathrm{inc}}(R)}{R}\right)^{2(n-1)} + (n-1)\left(\frac{R}{r^{\mathrm{inc}}(R)}\right)^2. \tag{2.8}$$

Remark 2.1. It is clear that, in general, there are infinitely many isochoric (non-radial) deformations of an annulus: e.g., in the case $n = 2$ consider deformations of the form

$$\mathbf{u}(\mathbf{x}) = \rho(R)\begin{pmatrix}\cos(\theta + \psi(R))\\ \sin(\theta + \psi(R))\end{pmatrix} \quad \text{for all } \mathbf{x} \in \overline{A}, \tag{2.9}$$

where (R, θ) are polar coordinates in the plane, $R = |\mathbf{x}|$, and $\rho \in C^1([a, b]; [a, b])$ and $\psi \in C^1([a, b])$ satisfy the boundary conditions $\rho(a) = \mu a$, $\rho(b) = \lambda b$, $\lambda^n b^n - \mu^n a^n = b^n - a^n$, $\psi(a) = 0$, and $\psi(b) = 2N\pi$ (corresponding to $N \in \mathbb{N}$ "twists" of the annulus). It follows that

$$\nabla\mathbf{u} = \begin{bmatrix}\rho' c - \rho\psi' s\\ \rho' s + \rho\psi' c\end{bmatrix} \otimes \frac{1}{R}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \frac{1}{R}\begin{bmatrix}-\rho s\\ \rho c\end{bmatrix} \otimes \frac{1}{R}\begin{bmatrix}-x_2\\ x_1\end{bmatrix},$$

where $c = c(R, \theta) := \cos(\theta + \psi(R))$ and $s = s(R, \theta) := \sin(\theta + \psi(R))$, and consequently

$$|\nabla\mathbf{u}|^2 = \mathrm{tr}\left[(\nabla\mathbf{u})(\nabla\mathbf{u})^{\mathrm{T}}\right] = \left(\frac{\rho}{R}\right)^2 + (\rho')^2 + (\rho\psi')^2 \tag{2.10}$$

and

$$(\det\nabla\mathbf{u})^2 = \det\left[(\nabla\mathbf{u})(\nabla\mathbf{u})^{\mathrm{T}}\right] = \left[\rho'\frac{\rho}{R}\right]^2.$$

Hence, for isochoric maps,

$$\rho(R) = \left(R^2 + (\lambda^2 - 1)b^2)\right)^{\frac{1}{2}} \tag{2.11}$$

and so (2.9) is a map of the annulus satisfying (1.6) for any choice of ψ and N.

3. Symmetry of Energy Minimising Deformations in Tension.

Let $\mathbf{u} \in C^1(\overline{A};\mathbb{R}^3)$ be any isochoric deformation that satisfies[4] $(1.6)_1$ for some $\mu \geq 1$. Our aim is to prove, in particular, that the energy functional (1.3) satisfies

$$E(\mathbf{u}) \geq E(\mathbf{u}_\mu^{\mathbf{rad}})$$

for any polyconvex stored-energy function W of the form

$$W(\mathbf{F}) = \Phi(|\mathbf{F}|^2,\ |\mathrm{adj}\,\mathbf{F}|), \tag{3.1}$$

where $\Phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is convex and monotone increasing in each of its arguments.

To present the main ideas in our proof we present our results first for the Dirichlet integral $W(\mathbf{F}) = |\mathbf{F}|^2$ (so that $\Phi(x, y) = x$ in (3.1)) and secondly for the case $W(\mathbf{F}) = |\,\mathrm{adj}\,\mathbf{F}|$ (so that $\Phi(x, y) = y$). To present the key ideas and simplify the technical details we will assume that $\mathbf{u} \in C^1(\overline{A};\mathbb{R}^3)$, however, the proofs easily generalise to wider classes of deformations $\mathbf{u}$ in the Sobolev space $W^{1,p}(A;\mathbb{R}^3)$, $p > 3$. (See section 5.)

Definition 3.1. For the displacement problem and the mixed problem where the outer boundary is left free we let $\mu > 0$ and define the set of admissible deformations by

$$\mathcal{A}_\mu^I = \left\{\mathbf{u} \in C^1(\overline{A};\mathbb{R}^3) : \det \nabla\mathbf{u} \equiv 1,\ \mathbf{u} \text{ is one-to-one},\ \ \mathbf{u}(\mathbf{x}) = \mu\mathbf{x} \text{ for } \mathbf{x} \in \partial A_I\right\}.$$

For the mixed problem where the inner boundary is left free we let $\lambda > 0$ and define the set of admissible deformations by

$$\mathcal{A}_\lambda^o = \left\{\mathbf{u} \in C^1(\overline{A};\mathbb{R}^3) : \det \nabla\mathbf{u} \equiv 1,\ \mathbf{u} \text{ is one-to-one},\ \ \mathbf{u}(\mathbf{x}) = \lambda\mathbf{x} \text{ for } \mathbf{x} \in \partial A_o\right\}.$$

For simplicity of exposition we present all of our results for $\mathbf{u} \in \mathcal{A}_\mu^I$ and note that the proof for $\mathbf{u} \in \mathcal{A}_\lambda^o$ is similar.

The next proposition will be central to the arguments in this paper and shows that the radial map (2.3)–(2.4) has the property that, amongst all maps in $\mathcal{A}_\mu^I$, it minimises the deformed area of each sphere S_R, $R \in [a, b]$.

Proposition 3.2. *Let $\mu > 0$ and $\mathbf{u} \in \mathcal{A}_\mu^I$. Then, for each $R \in [a, b]$,*

$$\mathrm{area}(\mathbf{u}(S_R)) \geq \mathrm{area}(\mathbf{u}_\mu^{\mathbf{rad}}(S_R)),$$

where $\mathbf{u}_\mu^{\mathbf{rad}}$ is given by (2.3)–(2.4) *and S_R is the sphere of radius R centred at the origin. Moreover, the above inequality is strict at any R for which $\mathbf{u}(S_R)$ is not a sphere.*

[4] By Remark 2.1 there are an infinite number of such deformations.

Proof. Fix $R \in [a, b]$. Let $\mathbf{u} \in C^2(\overline{A}; \mathbb{R}^3)$. Then the well-known divergence identity

$$\det \nabla \mathbf{u} = \frac{\partial}{\partial x^\alpha}\left(\frac{1}{3}\mathrm{u}^i (\mathrm{adj}\, \nabla \mathbf{u})_i^\alpha\right) = \mathrm{div}\left(\frac{1}{3}(\mathrm{adj}\, \nabla \mathbf{u})\mathbf{u}\right),$$

when integrated over $B_R \backslash B_a$, yields[5]

$$\int_{B_R \backslash B_a} \det \nabla \mathbf{u}\, d\mathbf{x} = \int_{S_R} \frac{1}{3}\mathbf{u} \cdot (\mathrm{adj}\, \nabla \mathbf{u})^{\mathrm{T}}\mathbf{n} \; - \int_{S_a} \frac{1}{3}\mathbf{u} \cdot (\mathrm{adj}\, \nabla \mathbf{u})^{\mathrm{T}}\mathbf{n}, \tag{3.2}$$

where $\mathbf{n} = (n^1, n^2, n^3)$ is the outward unit normal to $B_R \backslash B_a$. However, C^2 is dense in C^1 and so the bounded convergence theorem implies that (3.2) is satisfied by all $\mathbf{u} \in C^1(\overline{A}; \mathbb{R}^3)$ and hence, in particular, for all $\mathbf{u} \in \mathcal{A}_\mu^I$.

Now fix $\mathbf{u} \in \mathcal{A}_\mu^I$. The proof will follow from the classical isoperimetric inequality once we prove that the volumes of the regions enclosed by the surfaces $\mathbf{u}(S_R)$ and $\mathbf{u}_\mu^{\mathbf{rad}}(S_R)$ are the same. To this end we first note that, since $\det \nabla \mathbf{u} \equiv 1$,

$$\int_{B_R \backslash B_a} \det \nabla \mathbf{u}\, d\mathbf{x} = \frac{4}{3}\pi(R^3 - a^3) \tag{3.3}$$

while the boundary condition $\mathbf{u}(\mathbf{x}) = \mu \mathbf{x}$ for $\mathbf{x} \in S_a$ together with (2.3)–(2.6) implies that the boundary integral on S_a is given by

$$\int_{S_a} \frac{1}{3}\mathbf{u} \cdot (\mathrm{adj}\, \nabla \mathbf{u})^{\mathrm{T}}\mathbf{n} = \frac{4}{3}\pi\mu^3 a^3, \tag{3.4}$$

and hence is constant for all maps $\mathbf{u} \in \mathcal{A}_\mu^I$. In particular, applying (3.2)–(3.4) to $\mathbf{u}_\mu^{\mathbf{rad}}$ yields

$$\int_{S_R} \frac{1}{3}\mathbf{u} \cdot (\mathrm{adj}\, \nabla \mathbf{u})^{\mathrm{T}}\mathbf{n} = \int_{S_R} \frac{1}{3}\left(\mathbf{u}_\mu^{\mathbf{rad}}\right) . \left(\mathrm{adj}\, \nabla \mathbf{u}_\mu^{\mathbf{rad}}\right)^{\mathrm{T}}\mathbf{n}, \tag{3.5}$$

where $\mathbf{n}$ is now the outward unit normal to B_R. Next, in view of (2.3) and (2.6), the right-hand integral in (3.5) is given by

$$\int_{S_R} \frac{1}{3}\left(\mathbf{u}_\mu^{\mathbf{rad}}\right) . \left(\mathrm{adj}\, \nabla \mathbf{u}_\mu^{\mathbf{rad}}\right)^{\mathrm{T}}\mathbf{n} = \frac{4}{3}\pi\left[r_\mu^{\mathrm{inc}}(R)\right]^3, \tag{3.6}$$

which is the volume of the region bounded by the spherical surface $\mathbf{u}_\mu^{\mathbf{rad}}(S_R)$.

Finally, the Jordan separation theorem implies that $\mathbf{u}(S_R)$ divides $\mathbb{R}^3$ into two open regions, one bounded and one unbounded. Setting $\mathbf{y} = \mathbf{u}(\mathbf{x})$ and using the change of variables formula for surface integrals[6] in the left-hand integral in (3.5) we obtain

$$\int_{S_R} \frac{1}{3}\mathbf{u} \cdot (\mathrm{adj}\, \nabla \mathbf{u})^{\mathrm{T}}\mathbf{n} = \int_{\mathbf{u}(S_R)} \frac{1}{3}\,\mathbf{y} \cdot \mathbf{N}, \tag{3.7}$$

[5] In order to simplify our presentation we drop the surface measure, dS, from integrals over S_R.

[6] Since our deformations preserve orientation (3.7) is satisfied with $\mathbf{N}(\mathbf{u}(\mathbf{y})) = [\mathbf{A}(\mathbf{x})]^{\mathrm{T}}\mathbf{n}(\mathbf{x})/|[\mathbf{A}(\mathbf{x})]^{\mathrm{T}}\mathbf{n}(\mathbf{x})|$, $\mathbf{A}(\mathbf{x}) = \mathrm{adj}\, \nabla \mathbf{u}(\mathbf{x})$. However, (3.7) is also valid for smooth injective mappings that reverse orientation; in this case the outward unit normal $\mathbf{N}$ is given by minus this quantity.

where $\mathbf{N}$ is the outward unit normal to the C^1 surface $\mathbf{u}(S_R)$, that is, where $\mathbf{N}$ points into the unbounded region. Consequently, an application of the divergence theorem to the right-hand side of (3.7) shows that it is equal to the volume of the bounded region enclosed by $\mathbf{u}(S_R)$. Therefore, by (3.5)–(3.7) it follows that $\mathbf{u}(S_R)$ and $\mathbf{u}_\mu^{\mathbf{rad}}(S_R)$ both enclose the same volume; the claims of the lemma, both the area inequality and its strictness when $\mathbf{u}(S_R)$ is not a sphere, now follow from the classical isoperimetric inequality. □

3.1. The case $W(\mathbf{F}) = |\mathbf{F}|^2$.

Lemma 3.3. *Let $\mu > 0$, $\mathbf{u} \in \mathcal{A}_\mu^I$, and $R \in [a, b]$. At each point $\mathbf{x} \in S_R$ let $\mathbf{n}, \mathbf{t_1}, \mathbf{t_2}$ denote a right-handed orthonormal basis with $\mathbf{n} = \frac{\mathbf{x}}{|\mathbf{x}|}$. Then*

$$|\nabla\mathbf{u}(\mathbf{x})|^2 = \left|\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\right|^2 + \left|\frac{\partial\mathbf{u}}{\partial\mathbf{t_1}}\right|^2 + \left|\frac{\partial\mathbf{u}}{\partial\mathbf{t_2}}\right|^2 \quad \text{for } \mathbf{x} \in S_R.$$

Proof. This is a standard consequence of the invariance of the Dirichlet integral under orthogonal changes of coordinates that follows easily on writing

$$\begin{aligned}\nabla\mathbf{u} &= \nabla\mathbf{u}\,[\mathbf{n}\otimes\mathbf{n} + \mathbf{t_1}\otimes\mathbf{t_1} + \mathbf{t_2}\otimes\mathbf{t_2}] \\ &= \frac{\partial\mathbf{u}}{\partial\mathbf{n}}\otimes\mathbf{n} + \frac{\partial\mathbf{u}}{\partial\mathbf{t_1}}\otimes\mathbf{t_1} + \frac{\partial\mathbf{u}}{\partial\mathbf{t_2}}\otimes\mathbf{t_2}\end{aligned}$$

and then observing that

$$\begin{aligned}|\nabla\mathbf{u}|^2 = \operatorname{tr}\left((\nabla\mathbf{u})(\nabla\mathbf{u})^{\mathrm{T}}\right) &= \operatorname{tr}\left(\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\otimes\frac{\partial\mathbf{u}}{\partial\mathbf{n}} + \frac{\partial\mathbf{u}}{\partial\mathbf{t_1}}\otimes\frac{\partial\mathbf{u}}{\partial\mathbf{t_1}} + \frac{\partial\mathbf{u}}{\partial\mathbf{t_2}}\otimes\frac{\partial\mathbf{u}}{\partial\mathbf{t_2}}\right) \\ &= \left|\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\right|^2 + \left|\frac{\partial\mathbf{u}}{\partial\mathbf{t_1}}\right|^2 + \left|\frac{\partial\mathbf{u}}{\partial\mathbf{t_2}}\right|^2,\end{aligned}$$

as claimed. □

Lemma 3.4. *Let $\mu > 0$ and $\mathbf{u} \in \mathcal{A}_\mu^I$. Then for each $\mathbf{x} \in S_R$, $R \in [a, b]$, we have*

$$|\nabla\mathbf{u}|^2 \geq \frac{1}{\left|\frac{\partial\mathbf{u}}{\partial\mathbf{t_1}}\times\frac{\partial\mathbf{u}}{\partial\mathbf{t_2}}\right|^2} + 2\left|\frac{\partial\mathbf{u}}{\partial\mathbf{t_1}}\times\frac{\partial\mathbf{u}}{\partial\mathbf{t_2}}\right|. \tag{3.8}$$

Proof. It follows from Lemma 3.3 and the arithmetic-geometric inequality that

$$|\nabla\mathbf{u}|^2 \geq \left|\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\right|^2 + 2\left|\frac{\partial\mathbf{u}}{\partial\mathbf{t_1}}\right|\left|\frac{\partial\mathbf{u}}{\partial\mathbf{t_2}}\right| \geq \left|\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\right|^2 + 2\left|\frac{\partial\mathbf{u}}{\partial\mathbf{t_1}}\times\frac{\partial\mathbf{u}}{\partial\mathbf{t_2}}\right|. \tag{3.9}$$

Next note that since $\mathbf{u}$ is isochoric the Cauchy-Schwarz inequality implies

$$1 = \det\nabla\mathbf{u} = \frac{\partial\mathbf{u}}{\partial\mathbf{n}}\cdot\left(\frac{\partial\mathbf{u}}{\partial\mathbf{t_1}}\times\frac{\partial\mathbf{u}}{\partial\mathbf{t_2}}\right) \leq \left|\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\right|\left|\frac{\partial\mathbf{u}}{\partial\mathbf{t_1}}\times\frac{\partial\mathbf{u}}{\partial\mathbf{t_2}}\right|. \tag{3.10}$$

If we combine this with (3.9) it follows that (3.8) holds. □

Lemma 3.5. *Let*

$$g(t) = \frac{1}{t^2} + 2t \text{ for } t \in (0, \infty)$$

then g is convex on $(0, \infty)$ and monotone increasing for $t \geq 1$.

Remark 3.6. Note that the function $\sqrt{g(t)}$, which corresponds to the choice of stored energy function $W(\mathbf{F}) = |\mathbf{F}|$ (rather than $W(\mathbf{F}) = |\mathbf{F}|^2$), is not convex for large values of t.

Lemma 3.7. *Let $\mu > 0$ and $\mathbf{u} \in \mathcal{A}^I_\mu$. Then, for each $R \in [a, b]$,*

$$\int_{S_R} |\nabla \mathbf{u}|^2 \geq \int_{S_R} g\left(\left|\frac{\partial \mathbf{u}}{\partial \mathbf{t_1}} \times \frac{\partial \mathbf{u}}{\partial \mathbf{t_2}}\right|\right) \geq 4\pi R^2 g\left(\frac{\int_{S_R}\left|\frac{\partial \mathbf{u}}{\partial \mathbf{t_1}} \times \frac{\partial \mathbf{u}}{\partial \mathbf{t_2}}\right|}{4\pi R^2}\right). \tag{3.11}$$

Proof. If we integrate (3.8) over S_R the result will follow from Jensen's inequality since g is convex by Lemma 3.5. □

Lemma 3.8. *Let $\mu \geq 1$ and $\mathbf{u} \in \mathcal{A}^I_\mu$. Then, for each $R \in [a, b]$,*

$$\int_{S_R} \left|\frac{\partial \mathbf{u}}{\partial \mathbf{t_1}} \times \frac{\partial \mathbf{u}}{\partial \mathbf{t_2}}\right| \geq \int_{S_R} \left|\frac{\partial \mathbf{u}^{\mathbf{rad}}_\mu}{\partial \mathbf{t_1}} \times \frac{\partial \mathbf{u}^{\mathbf{rad}}_\mu}{\partial \mathbf{t_2}}\right| = 4\pi \left[r^{\mathrm{inc}}_\mu(R)\right]^2 \geq 4\pi R^2$$

and hence

$$\frac{\int_{S_R}\left|\frac{\partial \mathbf{u}}{\partial \mathbf{t_1}} \times \frac{\partial \mathbf{u}}{\partial \mathbf{t_2}}\right|}{4\pi R^2} \geq \frac{\int_{S_R}\left|\frac{\partial \mathbf{u}^{\mathbf{rad}}_\mu}{\partial \mathbf{t_1}} \times \frac{\partial \mathbf{u}^{\mathbf{rad}}_\mu}{\partial \mathbf{t_2}}\right|}{4\pi R^2} \geq 1.$$

Proof. This result is an immediate consequence of (2.3), (2.4), and Proposition 3.2 since, for any $\mathbf{v} \in \mathcal{A}^I_\mu$ and $R \in [a, b]$,

$$\int_{S_R} \left|\frac{\partial \mathbf{v}}{\partial \mathbf{t_1}} \times \frac{\partial \mathbf{v}}{\partial \mathbf{t_2}}\right| = \mathrm{area}(\mathbf{v}(S_R)) \quad \text{and} \quad \left[r^{\mathrm{inc}}_\mu(R)\right]^3 = R^3 + a^3(\mu^3 - 1) \geq R^3$$

for $\mu \geq 1$. □

Theorem 3.9. *Let $\mu \geq 1$ and $\mathbf{u} \in \mathcal{A}^I_\mu$. Then*

$$\int_A |\nabla \mathbf{u}|^2 d\mathbf{x} \geq \int_A |\nabla \mathbf{u}^{\mathbf{rad}}_\mu|^2 d\mathbf{x}. \tag{3.12}$$

Proof. If we integrate (3.11) with respect to R and make use of Lemmas 3.5 and 3.8 we find that

$$\int_A |\nabla \mathbf{u}|^2 d\mathbf{x} \geq \int_a^b 4\pi R^2 g\left(\frac{\int_{S_R}\left|\frac{\partial \mathbf{u}}{\partial \mathbf{t_1}} \times \frac{\partial \mathbf{u}}{\partial \mathbf{t_2}}\right|}{4\pi R^2}\right) dR$$
$$\geq \int_a^b 4\pi R^2 g\left(\frac{\int_{S_R}\left|\frac{\partial \mathbf{u}_\mu^{\mathbf{rad}}}{\partial \mathbf{t_1}} \times \frac{\partial \mathbf{u}_\mu^{\mathbf{rad}}}{\partial \mathbf{t_2}}\right|}{4\pi R^2}\right) dR. \tag{3.13}$$

Next note that by (2.1) and (2.7)

$$g\left(\frac{\int_{S_R}\left|\frac{\partial \mathbf{u}_\mu^{\mathbf{rad}}}{\partial \mathbf{t_1}} \times \frac{\partial \mathbf{u}_\mu^{\mathbf{rad}}}{\partial \mathbf{t_2}}\right|}{4\pi R^2}\right) = g\left(\left(\frac{r_\mu^{\text{inc}}(R)}{R}\right)^2\right) = |\nabla \mathbf{u}_\mu^{\mathbf{rad}}|^2,$$

where $\mathbf{u}_\mu^{\mathbf{rad}}$ and r_μ^{inc} are given by (2.3) and (2.4). If we multiply the above equation by $4\pi R^2$ and integrate with respect to R we find, with the aid of (3.13), that (3.12) is satisfied. $\square$

Remark 3.10. In the two-dimensional case, $n = 2$, the result corresponding to Theorem 3.9 is that

$$\int_A |\nabla \mathbf{u}|\, d\mathbf{x} \geq \int_A |\nabla \mathbf{u}_\mu^{\mathbf{rad}}|\, d\mathbf{x} \quad \text{for all } \mathbf{u} \in \mathcal{A}_\mu^I,$$

where $\mathbf{u}_\mu^{\mathbf{rad}}$ is given by (2.3)–(2.4) with $n = 2$. This follows by analogous arguments to the case $n = 3$ with the following modifications:

1. For $n = 2$, the result corresponding to Proposition 3.2 is that $\mathbf{u}_\mu^{\mathbf{rad}}$ minimises the deformed length of each circle S_R for each $R \in [a, b]$.
2. For $n = 2$, the estimate corresponding to Lemma 3.4 is that for each $\mathbf{x} \in S_R$ and $R \in [a, b]$
$$|\nabla \mathbf{u}| \geq \left(\left|\frac{\partial \mathbf{u}}{\partial \mathbf{t}}\right|^{-2} + \left|\frac{\partial \mathbf{u}}{\partial \mathbf{t}}\right|^2\right)^{\frac{1}{2}}.$$
3. The convex function $g(t)$ in Lemma 3.5 is replaced by the convex function
$$\tilde{g}(t) = \left(t^{-2} + t^2\right)^{\frac{1}{2}} \quad \text{for } t \in (0, \infty).$$

3.2. The case $W(\mathbf{F}) = |\operatorname{adj} \mathbf{F}|$.

In this section we derive similar estimates to those obtained in the previous section but this time for the energy function $W(\mathbf{F}) = |\operatorname{adj} \mathbf{F}|$. We will make use of the following standard vector identities which are stated together in the following lemma for convenience.

Lemma 3.11. *Let $\mathbf{n}, \mathbf{t_1}, \mathbf{t_2}$ be a right-handed orthonormal set of vectors in $\mathbb{R}^3$. Then:*

(i) Let $\mathbf{A} \in M_1^{3\times 3}$ then $\operatorname{adj}(\operatorname{adj}\mathbf{A}) = \mathbf{A}$;

(ii) Let $\mathbf{G} \in M^{3\times 3}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ then $\mathbf{Ga} \times \mathbf{Gb} = (\operatorname{adj}\mathbf{G})^{\mathrm{T}}(\mathbf{a} \times \mathbf{b})$;

(iii) Since $\mathbf{n} = \mathbf{t_1} \times \mathbf{t_2}$ it follows from (ii) that, for any $\mathbf{G} \in M^{3\times 3}$,

$$|(\mathbf{G}\mathbf{t_1}) \times (\mathbf{G}\mathbf{t_2})| = \left|(\operatorname{adj}\mathbf{G})^{\mathrm{T}}\, \mathbf{n}\right|.$$

Lemma 3.12. *Let $\mu > 0$ and $\mathbf{u} \in \mathcal{A}_\mu^I$. Then for each $\mathbf{x} \in S_R$, $R \in [a, b]$, we have*

$$|\operatorname{adj}\nabla\mathbf{u}|^2 \geq \left|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right|^2 + \frac{2}{\left|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right|}. \tag{3.14}$$

Proof. Writing $\mathbf{G} := \operatorname{adj}\nabla\mathbf{u}$ we note that

$$\begin{aligned} |\mathbf{G}|^2 = \left|\mathbf{G}^{\mathrm{T}}\right|^2 &= \left|\mathbf{G}^{\mathrm{T}}\mathbf{n}\right|^2 + \left|\mathbf{G}^{\mathrm{T}}\mathbf{t}_1\right|^2 + \left|\mathbf{G}^{\mathrm{T}}\mathbf{t}_2\right|^2 \\ &\geq \left|\mathbf{G}^{\mathrm{T}}\mathbf{n}\right|^2 + 2\left|\left(\mathbf{G}^{\mathrm{T}}\mathbf{t}_1\right) \times \left(\mathbf{G}^{\mathrm{T}}\mathbf{t}_2\right)\right|. \end{aligned} \tag{3.15}$$

Next, observe that, by (i) and (iii) of Lemma 3.11,

$$\left((\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{t}_1\right) \times \left((\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{t}_2\right) = (\nabla\mathbf{u})\mathbf{n}. \tag{3.16}$$

Finally, by the Cauchy-Schwarz inequality and the fact that $\det\nabla\mathbf{u} = 1$,

$$1 = |\mathbf{n}|^2 = \left|(\nabla\mathbf{u})\mathbf{n} \cdot (\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right| \leq \left|(\nabla\mathbf{u})\mathbf{n}\right| \left|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right|. \tag{3.17}$$

The result (3.14) now follows on combining (3.15)–(3.17). □

Lemma 3.13. *Define*

$$h(t) = \sqrt{t^2 + \frac{2}{t}} \quad \text{for } t \in (0, \infty).$$

Then h is convex on $(0, \infty)$ and monotone increasing for $t \geq 1$.

Theorem 3.14. *Let $\mu \geq 1$ and $\mathbf{u} \in \mathcal{A}_\mu^I$. Then*

$$\int_A |\operatorname{adj}\nabla\mathbf{u}|\, d\mathbf{x} \geq \int_A |\operatorname{adj}\nabla\mathbf{u}_\mu^{\mathbf{rad}}|\, d\mathbf{x}. \tag{3.18}$$

Proof. The proof is analogous to that of Theorem 3.9. By Lemmas 3.12, 3.13, and Jensen's inequality

$$\int_a^b \int_{S_R} |\operatorname{adj} \nabla \mathbf{u}|\, d\mathbf{x} \geq \int_a^b 4\pi R^2 h \left(\frac{\int_{S_R} \left|(\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}} \mathbf{n}\right|}{4\pi R^2} \right) dR$$

$$\geq \int_a^b 4\pi R^2 h \left(\frac{\int_{S_R} \left|\left(\operatorname{adj} \nabla \mathbf{u}_\mu^{\mathbf{rad}}\right)^{\mathrm{T}} \mathbf{n}\right|}{4\pi R^2} \right) dR,$$

where the last inequality follows from the monotonicity of h and Proposition 3.2. We have also used the fact that $\int_{S_R} |(\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}} \mathbf{n}|$ is the area of $\mathbf{u}\,(S_R)$ since, by Lemma 3.11(iii),

$$\int_{S_R} \left|(\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}} \mathbf{n}\right| = \int_{S_R} \left| \frac{\partial \mathbf{u}}{\partial \mathbf{t_1}} \times \frac{\partial \mathbf{u}}{\partial \mathbf{t_2}} \right| = \operatorname{area}(\mathbf{u}(S_R)).$$

Next, note that by (2.6) and (2.8),

$$h \left(\frac{\int_{S_R} \left|\left(\operatorname{adj} \nabla \mathbf{u}_\mu^{\mathbf{rad}}\right)^{\mathrm{T}} \mathbf{n}\right|}{4\pi R^2} \right) = h \left(\left(\frac{r_\mu^{\mathrm{inc}}(R)}{R} \right)^2 \right) = |\operatorname{adj} \nabla \mathbf{u}_\mu^{\mathbf{rad}}|,$$

where r_μ^{inc} and $\mathbf{u}_\mu^{\mathbf{rad}}$ are related by (2.3). If we multiply the above equation by $4\pi R^2$ and integrate with respect to R we arrive at (3.18) as claimed. □

3.3. The general case: $W(\mathbf{F}) = \Phi(|\mathbf{F}|^2, |\operatorname{adj} \mathbf{F}|)$.

Now suppose that $W(\mathbf{F}) = \Phi(|\mathbf{F}|^2, |\operatorname{adj} \mathbf{F}|)$. If we now combine the arguments of sections 3.1 and 3.2 we obtain the following result.

Theorem 3.15. *Let $\mu \geq 1$ and*

$$W(\mathbf{F}) = \Phi(|\mathbf{F}|^2, |\operatorname{adj} \mathbf{F}|),$$

where $\Phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is convex, and Φ is an increasing function in each of its arguments. Then, for any $\mathbf{u} \in \mathcal{A}_\mu^I$,

$$\int_A W(\nabla \mathbf{u})\, d\mathbf{x} \geq \int_A W(\nabla \mathbf{u}_\mu^{\mathbf{rad}})\, d\mathbf{x}.$$

Proof. We sketch the proof of this energy inequality since it is analogous to the proofs of Theorems 3.9 and 3.14: using Jensen's inequality, the monotonicity of Φ, and the isoperimetric inequality

$$\begin{aligned}
\int_A W(\nabla\mathbf{u})\,d\mathbf{x} &= \int_a^b \left(\int_{S_R} \Phi(|\nabla\mathbf{u}|^2, |\operatorname{adj}\nabla\mathbf{u}|)\right) dR \\
&\geq \int_a^b 4\pi R^2 \Phi\left(g\left(\frac{\int_{S_R}|\nabla\mathbf{u}|^2}{4\pi R^2}\right), h\left(\frac{\int_{S_R}|\operatorname{adj}\nabla\mathbf{u}|}{4\pi R^2}\right)\right) dR \\
&\geq \int_a^b 4\pi R^2\ \Phi\left(g\left(\frac{\int_{S_R}|\nabla\mathbf{u}_\mu^{\mathbf{rad}}|^2}{4\pi R^2}\right), h\left(\frac{\int_{S_R}|\operatorname{adj}\nabla\mathbf{u}_\mu^{\mathbf{rad}}|}{4\pi R^2}\right)\right) dR \\
&= \int_a^b 4\pi R^2\Phi\left(|\nabla\mathbf{u}_\mu^{\mathbf{rad}}|^2\ , |\operatorname{adj}\nabla\mathbf{u}_\mu^{\mathbf{rad}}|\right) dR = \int_A W(\nabla\mathbf{u}_\mu^{\mathbf{rad}})\,d\mathbf{x},
\end{aligned}$$

where g and h are given by Lemmas 3.5 and 3.13. □

Remark 3.16. The energy of $\mathbf{u}$ will be *strictly greater* than the energy of $\mathbf{u}_\mu^{\mathbf{rad}}$ if any of the inequalities used in our derivation is strict. In particular, in equation (3.17) (and (3.10))

$$\left|(\nabla\mathbf{u})\mathbf{n}\cdot(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right| \leq \left|(\nabla\mathbf{u})\mathbf{n}\right|\left|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right|$$

is a strict inequality unless the vectors $(\nabla\mathbf{u})\mathbf{n}$ and $(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}$ are parallel. In this case it then follows that $\mathbf{n}$ is an eigenvector of $(\nabla\mathbf{u})^{\mathrm{T}}\nabla\mathbf{u}$.

Remark 3.17. The corresponding result in the two-dimensional case, $n=2$, is that if

$$W(\mathbf{F}) = \Phi(|\mathbf{F}|),$$

where $\Phi:\mathbb{R}^+\to\mathbb{R}$ is convex and monotone increasing, then for any $\mathbf{u}\in\mathcal{A}_\mu^I$

$$\int_A W(\nabla\mathbf{u})\,d\mathbf{x} \geq \int_A W(\nabla\mathbf{u}_\mu^{\mathbf{rad}})\,d\mathbf{x}.$$

The proof of this follows exactly as in Theorem 3.15 on noting the results of Remark 3.10.

4. Symmetry of Energy Minimising Deformations in Compression.

In this section we will show that, for the pure displacement problem, the results in the previous section can be extended to compression (i.e., to the case $\mu\in(0,1)$).

Definition 4.1. Fix $\mu>0$ and define the set of admissible deformations by

$$\mathcal{A}_\mu^D = \left\{\ \mathbf{u}\in C^1(\overline{A};\mathbb{R}^3) : \det\nabla\mathbf{u}\equiv 1,\ \begin{array}{l}\mathbf{u}(\mathbf{x}) = \mu\mathbf{x} \text{ for } \mathbf{x}\in\partial A_I, \\ \mathbf{u}(\mathbf{x}) = \lambda\mathbf{x} \text{ for } \mathbf{x}\in\partial A_o\end{array}\right\},$$

where $\lambda>0$ satisfies $\lambda^3 b^3 - \mu^3 a^3 = b^3 - a^3$.

We note results from degree theory (see, e.g., [9] or [21]) imply that the image of A under any continuous, one-to-one map that satisfies the given boundary conditions is the annulus

$$A^* = \left\{ \mathbf{x} \in \mathbb{R}^3 : \mu a < |\mathbf{x}| < \lambda b \right\},$$

and, moreover, that such a map is open and satisfies $\mathbf{u}(A) = A^*$. The following result then follows from degree theory and the inverse function theorem.[7]

Proposition 4.2. *Let $\mu > 0$ and $\mathbf{u} \in \mathcal{A}_\mu^D$. Then $\mathbf{u}$ is one-to-one and satisfies $\mathbf{u}(A) = A^*$. Moreover, $\mathbf{u}$ has an inverse $\mathbf{v} = \mathbf{v_u} := \mathbf{u}^{-1} \in C^1(\overline{A^*}; \overline{A})$; this inverse is one-to-one and satisfies $\mathbf{v}(A^*) = A$, $\det \nabla \mathbf{v} \equiv 1$, and*

$$\mathbf{v}(\mathbf{y}) = \mu^{-1}\mathbf{y} \text{ for } \mathbf{y} \in \partial A_I^*, \qquad \mathbf{v}(\mathbf{y}) = \lambda^{-1}\mathbf{y} \text{ for } \mathbf{y} \in \partial A_o^*,$$

where ∂A_I^* and ∂A_o^* are the inner and outer boundaries of A^*, respectively.

In order to compare energies in compression we follow Ball [2, pp. 210–211] and change variables to the deformed configuration.

Proposition 4.3. *Let $\mu > 0$ and $\mathbf{u} \in \mathcal{A}_\mu^D$. Suppose that $\Psi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is continuous. Then $\mathbf{u}$ and its inverse $\mathbf{v}$ satisfy*

$$\int_A \Psi\left(|\nabla_\mathbf{x}\mathbf{u}(\mathbf{x})|, |\operatorname{adj} \nabla_\mathbf{x}\mathbf{u}(\mathbf{x})|^2\right) d\mathbf{x} = \int_{A^*} \Psi\left(|\operatorname{adj} \nabla_\mathbf{y}\mathbf{v}(\mathbf{y})|, |\nabla_\mathbf{y}\mathbf{v}(\mathbf{y})|^2\right) d\mathbf{y}.$$

Proof. Let $\mathbf{u} \in \mathcal{A}_\mu^D$ with inverse $\mathbf{v}$. Then $\mathbf{x} \mapsto \Psi\left(|\nabla\mathbf{u}(\mathbf{x})|, |\operatorname{adj} \nabla\mathbf{u}(\mathbf{x})|^2\right)$ is continuous and so the change of variables formula together with $\det \nabla \mathbf{v} \equiv 1$ yields

$$\int_A \Psi\left(|\nabla_\mathbf{x}\mathbf{u}(\mathbf{x})|, |\operatorname{adj} \nabla_\mathbf{x}\mathbf{u}(\mathbf{x})|^2\right) d\mathbf{x} = \int_{A^*} \Psi\left(|\nabla_\mathbf{x}\mathbf{u}(\mathbf{v}(\mathbf{y}))|, |\operatorname{adj} \nabla_\mathbf{x}\mathbf{u}(\mathbf{v}(\mathbf{y}))|^2\right) d\mathbf{y}. \tag{4.1}$$

Next, $(\mathbf{u} \circ \mathbf{v})(\mathbf{y}) = \mathbf{y}$ and hence $[\nabla_\mathbf{x}\mathbf{u}(\mathbf{v}(\mathbf{y}))][\nabla_\mathbf{y}\mathbf{v}(\mathbf{y})] = \mathbf{I}$. Therefore, since $\operatorname{adj} \nabla \mathbf{v} = [\nabla \mathbf{v}]^{-1}$ for any isochoric deformation,

$$\nabla_\mathbf{x}\mathbf{u}(\mathbf{v}(\mathbf{y})) = \operatorname{adj}\big[\nabla_\mathbf{y}\mathbf{v}(\mathbf{y})\big] \text{ for every } \mathbf{y} \in A^*.$$

Consequently, if we take the adjugate of both sides and apply Lemma 3.11(i) we find that

$$\operatorname{adj}\big[\nabla_\mathbf{x}\mathbf{u}(\mathbf{v}(\mathbf{y}))\big] = \nabla_\mathbf{y}\mathbf{v}(\mathbf{y}) \text{ for every } \mathbf{y} \in A^*.$$

The last two equations and (4.1) then yield the desired result. □

Since $\mu^{-1} \geq 1$ when $\mu \leq 1$ the results in the previous section together with Proposition 4.3 then yield the main result of this section.

[7] See, e.g., [6, Theorem 5.5-2] and recall that $\mathbf{u} \in C^1(\overline{A})$ means $\mathbf{u}$ is C^1 on a open set containing $\overline{A}$.

Theorem 4.4. *Let* $\mu \in (0,1]$ *and*

$$W(\mathbf{F}) = \Psi(|\mathbf{F}|, |\operatorname{adj}\mathbf{F}|^2),$$

where $\Psi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is convex and* Ψ *is an increasing function in each of its arguments. Then, for any* $\mathbf{u} \in \mathcal{A}_\mu^D$,

$$\int_A W(\nabla\mathbf{u})\, d\mathbf{x} \geq \int_A W(\nabla\mathbf{u}_\mu^{\mathbf{rad}})\, d\mathbf{x}.$$

Finally, Theorems 3.15 and 4.4 together yield the following result.

Corollary 4.5. *Let* $\mu > 0$ *and*

$$W(\mathbf{F}) = \Psi(|\mathbf{F}|^2, |\operatorname{adj}\mathbf{F}|^2),$$

where $\Psi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is convex and* Ψ *is an increasing function in each of its arguments. Then, for any* $\mathbf{u} \in \mathcal{A}_\mu^D$,

$$\int_A W(\nabla\mathbf{u})\, d\mathbf{x} \geq \int_A W(\nabla\mathbf{u}_\mu^{\mathbf{rad}})\, d\mathbf{x}.$$

Remark 4.6. The results in Section 3 rely on the idea that the image of each spherical shell centred at the origin in the reference configuration, $S_R \subset A$, prefers to retain its spherical shape in order to minimize the elastic energy. The results in this section instead use the property that the preimage of any spherical shell centred at the origin, in the deformed configuration, $S_r \subset A^*$, prefers to be the image of some spherical shell centred at the origin. This idea cannot be applied unless the deformed configuration is the union of such shells, which necessitates that the image of the annulus A be another annulus; thus the technique in this section is only applicable to the pure displacement problem.

5. Sobolev Deformations: Symmetry of Global Minimisers

We now generalize the results of sections 3 and 4 to allow for deformations given by the existence theory of Ball [1] (and subsequent generalisation in [5, 19, 24, 25]). In this section we restrict attention to the displacement boundary-value problem and the mixed problem with a free outer boundary. There are technical difficulties associated with the mixed problem with a free inner boundary: in particular, it is not clear to us that part (f) of Proposition 5.5 remains valid for this problem.[8]

Definition 5.1. Suppose that $p > 2$ and $\mu > 0$. For the pure displacement problem we define the set of admissible *Sobolev deformations* by

$$\mathcal{S}_\mu^p = \left\{ \mathbf{u} \in W^{1,p}(A;\mathbb{R}^3) : \det\nabla\mathbf{u} = 1 \ a.e., \ \begin{array}{l} \mathbf{u}(\mathbf{x}) = \mu\mathbf{x} \text{ for } \mathbf{x} \in \partial A_I, \\ \mathbf{u}(\mathbf{x}) = \lambda\mathbf{x} \text{ for } \mathbf{x} \in \partial A_o \end{array} \right\},$$

[8] However, this difficulty can be avoided by additional assumptions as to what constitutes a deformation, see, e.g., Henao [13].

where $\lambda > 0$ satisfies $\lambda^3 b^3 - \mu^3 a^3 = b^3 - a^3$ and $W^{1,p}(A;\mathbb{R}^3)$ denotes the usual Sobolev space of vector-valued functions $\mathbf{u} \in \mathcal{L}^p(A;\mathbb{R}^3)$, whose distributional derivative also lies in $\mathcal{L}^p$. If $p > 3$ then, by the Sobolev imbedding theorem, each $\mathbf{u} \in \mathcal{S}^p_\mu$ has a representative that is continuous[9] and so we may then assume that $\mathbf{u} \in W^{1,p}(A) \cap C^0(\overline{A})$.

For the mixed problem, where the outer boundary is free, we take $p > 3$ and define the set of admissible *Sobolev deformations* by

$$\widehat{\mathcal{S}}^p_\mu = \left\{ \mathbf{u} \in W^{1,p}(A) \cap C^0(\overline{A}) : \begin{array}{cc} \det \nabla\mathbf{u} = 1 \ a.e., & \mathbf{u}(\mathbf{x}) \notin B_{\mu a} \text{ for a.e. } \mathbf{x} \in A, \\ \operatorname{vol}(\mathbf{u}(A)) = \frac{4\pi(b^3-a^3)}{3}, & \mathbf{u}(\mathbf{x}) = \mu\mathbf{x} \text{ for } \mathbf{x} \in \partial A_I \end{array} \right\}.$$

Remark 5.2. The inclusion $\mathbf{u}(P) \subset B_{\mu a}$ for a set of positive measure P can occur when $\mathbf{u}$ is not isochoric (see, e.g. Figure 6 in [17]). However, for $p > 6$ ($p > n(n-1)$ in n-dimensions) isochoric mappings in $W^{1,p}$ are open, by a result of Villamor and Manfredi [26] (see, also [12, 15]). It follows that if, in addition, $\mathbf{u}$ is one-to-one a.e. then $\mathbf{u}(\mathbf{x}) \notin B_{\mu a}$ for a.e. $\mathbf{x} \in A$. When $p \in (3, 6]$ it is not clear to us if this containment constraint is then a consequence of our other hypotheses. Note that $n = 2$ yields $p > 2(2-1) = 2$ and so the constraint is not needed.

The next two results, which give invertibility properties of deformations, are due to Ball [3] and Ciarlet and Nečas [5], respectively (see also [9, Chapter 6]).

Proposition 5.3. *Let $p > 3$ and $\mu > 0$. Suppose that $\mathbf{u} \in \mathcal{S}^p_\mu$. Then*

(a) There exists a Lebesgue null set $\mathcal{N} \subset A$ such that $\mathbf{u}$ is one-to-one on $A \backslash \mathcal{N}$; and

(b) $\mathbf{u}(\overline{A}) = \overline{A^*}$.

Moreover, if in addition $\operatorname{adj} \nabla\mathbf{u} \in \mathcal{L}^q(A)$ *for some $q > 3$ then*

(c) $\mathbf{u}$ is one-to-one on $\overline{A}$;

(d) $\mathbf{u}(A) = A^*$; and

(e) $\mathbf{u}$ has an inverse $\mathbf{v} \in W^{1,q}(A^*) \cap C^0(\overline{A^*})$ that satisfies $\nabla_{\mathbf{y}}\mathbf{v}(\mathbf{y}) = [\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{v}(\mathbf{y}))]^{-1}$ for a.e. $\mathbf{y} \in A^*$, where $\nabla\mathbf{w}$ denotes the matrix of weak derivatives of a mapping $\mathbf{w}$.

Proposition 5.4. *Let $p > 3$ and $\mu > 0$. Suppose that $\mathbf{u} \in \widehat{\mathcal{S}}^p_\mu$. Then there exists a Lebesgue null set $\mathcal{N} \subset A$ such that $\mathbf{u}$ is one-to-one on $A \backslash \mathcal{N}$.*

Before proceeding further we note certain other key properties of such mappings.

Proposition 5.5. *Let $p > 2$ and suppose that $\mathbf{u} \in W^{1,p}(A;\mathbb{R}^3)$ is one-to-one a.e. and satisfies* $\det \nabla\mathbf{u} > 0$ *a.e. on A. Then for $\mathcal{L}^1$ a.e. $R \in (a, b)$,*

[9] For $p \in [2, 3]$ see [27, Theorem 2.3.2] and [19, 24, 25].

(a) $\mathbf{u}|_{S_R} \in W^{1,p}(S_R) \cap C^0(S_R)$;

(b) $\mathbf{u}(S_R)$ *is* $\mathcal{H}^2$ *measurable with* $\mathcal{H}^2(\mathbf{u}(S_R)) \leq \int_{S_R} |(\operatorname{adj} \nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}| \, d\mathcal{H}^2_{\mathbf{x}}$;

(c) $\partial^*(\operatorname{im}_T(\mathbf{u}, S_R))$ *is* $\mathcal{H}^2$ *measurable with* $\mathcal{H}^2(\partial^*(\operatorname{im}_{\mathrm{T}}(\mathbf{u}, S_R))) = \mathcal{H}^2(\mathbf{u}(S_R))$;

(d) $36\pi\left[\mathcal{L}^3(\operatorname{im}_{\mathrm{T}}(\mathbf{u}, S_R))\right]^2 \leq \left[\mathcal{H}^2(\partial^*(\operatorname{im}_{\mathrm{T}}(\mathbf{u}, S_R)))\right]^3$;

(e) *For any* $\mathbf{v} \in C^1(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} \operatorname{degree}(\mathbf{u}, S_R, \mathbf{y}) \operatorname{div} \mathbf{v}(\mathbf{y}) \, d\mathbf{y} = \int_{S_R} \mathbf{v}(\mathbf{u}(\mathbf{x})) \cdot (\operatorname{adj} \nabla\mathbf{u}(\mathbf{x}))^{\mathrm{T}}\mathbf{n}(\mathbf{x}) \, d\mathcal{H}^2_{\mathbf{x}}; \textit{and}$$

(f) *If* $p > 3$, $\mu > 0$, *and* $\mathbf{u} \in \mathcal{S}^p_\mu \cap C^0(\overline{A})$ *or* $\mathbf{u} \in \widehat{\mathcal{S}}^p_\mu$ *then each* $\mathbf{y} \in \mathbb{R}^3 \backslash \mathbf{u}(S_R)$ *satisfies* $\operatorname{degree}(\mathbf{u}, S_R, \mathbf{y}) = 1$ *or* $\operatorname{degree}(\mathbf{u}, S_R, \mathbf{y}) = 0$.

Here

$$\operatorname{im}_{\mathrm{T}}(\mathbf{u}, S_R) := \left\{\mathbf{y} \in \mathbb{R}^3 \backslash \mathbf{u}(S_R) : \operatorname{degree}(\mathbf{u}, S_R, \mathbf{y}) \neq 0\right\} \tag{5.1}$$

is the topological image of S_R *under* $\mathbf{u}$, $\mathcal{H}^2$ *denotes two-dimensional Hausdorff measure,* $\partial^*\Omega$ *denotes the reduced boundary*[10] *of* Ω, $\mathcal{L}^3$ *denotes three-dimensional Lebesgue measure, and* degree *denotes the Brouwer degree.*[11]

Remark 5.6. Proposition 5.5(a) is well known, see, e.g., [7, 9, 29]. Part (b) is due to Marcus and Mizel [16] (see also [8, 9]). Part (c) can be found in, e.g., the proof of Lemma 3.5 (steps 1–3) in [17]. Part (d) is a version of the classical isoperimetric inequality. It can be found in, for example, [7, p. 190] or [29, Theorem 5.4.3]; the given (dimensionally dependent) constant 36π can be found in Federer [8, pp. 275, 278]. Part (e) can be found in, e.g., [18, Proposition 2.1].

Proof of (f). Let $p > 3$ and $\mathbf{u} \in \mathcal{S}^p_\mu \cap C^0(\overline{A})$ or $\mathbf{u} \in \widehat{\mathcal{S}}^p_\mu$. Define an extension of $\mathbf{u}$ by

$$\mathbf{u}^e(\mathbf{x}) = \begin{cases} \mathbf{u}(\mathbf{x}), & \text{if } \mathbf{x} \in \overline{A}, \\ \mu\mathbf{x}, & \text{if } \mathbf{x} \in B_a = B_b \backslash \overline{A}. \end{cases}$$

Clearly, $\mathbf{u}^e \in W^{1,p}(B_b) \cap C^0(\overline{B}_b)$ and $\det \nabla\mathbf{u}^e > 0$ a.e. in B_b.

If $\mathbf{u} \in \mathcal{S}^p_\mu \cap C^0(\overline{A})$ then, in view of Proposition 5.3, $\mathbf{u}^e$ is one-to-one a.e. on B_b. If instead $\mathbf{u} \in \widehat{\mathcal{S}}^p_\mu$ the same conclusion follows from Proposition 5.4 together with the exclusion property $\mathbf{u}(\mathbf{x}) \notin B_{\mu a}$ for a.e. $\mathbf{x} \in A$. In either case it follows that $\mathbf{u}^e$ satisfies condition (INV) of Müller and Spector [17]. Thus, we can apply Lemma 3.5 in [17] to conclude that the degree only assumes the values zero and one. Finally, for $R \in (a, b)$ the functions $\mathbf{u}$ and $\mathbf{u}^e$ are equal, which establishes (f) as the degree only depends on the boundary values. $\square$

[10] See, e.g., Chapter 5 in either [7] or [29].
[11] See, e.g., [9] or [21].

We are now ready to prove the analogue of Proposition 3.2 in a Sobolev space.

Lemma 5.7. *Let $\mu > 0$, $p > 3$, and $\mathbf{u} \in \mathcal{S}^p_\mu$ (or $\widehat{\mathcal{S}}^p_\mu$). Then for almost every $R \in (a,b)$*

$$\mathcal{H}^2(\mathbf{u}(S_R)) \geq \mathcal{H}^2(\mathbf{u}^{\mathbf{rad}}_\mu(S_R)),$$

where $\mathbf{u}^{\mathbf{rad}}_\mu$ is given by (2.3)–(2.4).

Proof. We first note that, since $\mathbf{u}^{\mathbf{rad}}_\mu(S_R)$ is the sphere of radius $r^{\rm inc}_\mu(R)$,

$$36\pi \left[\mathcal{L}^3\left(B_{r^{\rm inc}_\mu(R)}\right)\right]^2 = \left[\mathcal{H}^2\left(\mathbf{u}^{\mathbf{rad}}_\mu(S_R)\right)\right]^3. \tag{5.2}$$

The desired result will follow from (5.2) and the isoperimetric inequality, Proposition 5.5(c,d), once we establish that the open set $\mathrm{im}_{\mathrm{T}}(\mathbf{u}, S_R)$ has the same volume as $B_{r^{\rm inc}_\mu(R)}$, i.e., $\frac{4}{3}\pi r^{\rm inc}_\mu(R)^3 = \frac{4}{3}\pi[R^3 + a^3(\mu^3 - 1)]$, by (2.4).

Now, we have previously shown that equation (3.2) is satisfied by all mappings $\mathbf{u} \in C^2(\overline{A};\mathbb{R}^3)$. If, in addition, such a $\mathbf{u}$ satisfies the displacement boundary condition $\mathbf{u}(\mathbf{x}) = \mu\mathbf{x}$ on S_a then $\mathbf{u}$ also satisfies (3.4). Combining (3.2) and (3.4) we conclude that

$$\int_{B_R\setminus B_a} \det \nabla\mathbf{u}\, d\mathbf{x} = \int_{S_R} \frac{1}{3}\mathbf{u}\cdot(\mathrm{adj}\,\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n} \ - \frac{4}{3}\pi\mu^3 a^3 \tag{5.3}$$

for every $\mathbf{u} \in C^2(\overline{A};\mathbb{R}^3)$ that satisfies $\mathbf{u}(\mathbf{x}) = \mu\mathbf{x}$ on S_a.

However, the boundary is smooth and so $C^2(\overline{A};\mathbb{R}^3)$ is dense in $W^{1,p}(A)$. Now let $\mathbf{u} \in W^{1,p}(A)$ and consider a sequence of C^2 mappings $\mathbf{u}_n \to \mathbf{u}$ strongly in $W^{1,p}(A)$. Then, $p > 3$ yields $\mathbf{u}_n \to \mathbf{u}$ uniformly on A, by the Sobolev imbedding theorem, and $\det\nabla\mathbf{u}_n \to \det\nabla\mathbf{u}$ strongly in $\mathcal{L}^1(A)$. Next, by Proposition 5.5(a) together with Fubini's theorem, for $\mathcal{L}^1$ a.e. $R \in (a,b)$, there is a subsequence (not relabeled) that satisfies $\mathbf{u}_n \to \mathbf{u}$ strongly in $W^{1,p}(S_R)$ and consequently $\mathrm{adj}\,\nabla\mathbf{u}_n \to \mathrm{adj}\,\nabla\mathbf{u}$ strongly in $\mathcal{L}^1(S_R)$. We therefore conclude that, for a.e. $R \in (a,b)$, (5.3) is satisfied by all $\mathbf{u} \in W^{1,p}(A;\mathbb{R}^3)$ that obey $\mathbf{u}(\mathbf{x}) = \mu\mathbf{x}$ on S_a and, in particular, for all $\mathbf{u} \in \mathcal{S}^p_\mu$ (or $\widehat{\mathcal{S}}^p_\mu$).

We now note that, by (5.1) and Proposition 5.5(e,f), with $\mathbf{v}(\mathbf{y}) = \mathbf{y}/3$,

$$\mathcal{L}^3\left(\mathrm{im}_{\mathrm{T}}(\mathbf{u}, S_R)\right) = \frac{1}{3}\int_{S_R} \mathbf{u}\cdot(\mathrm{adj}\,\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\, d\mathcal{H}^2_{\mathbf{x}}. \tag{5.4}$$

Finally, since $\det\nabla\mathbf{u} = 1$ a.e.,

$$\int_{B_R\setminus B_a} \det\nabla\mathbf{u}\, d\mathbf{x} = \frac{4}{3}\pi(R^3 - a^3), \tag{5.5}$$

and the desired result follows from (5.3)–(5.5). $\square$

The next theorem then extends the results in section 3 to such Sobolev deformations.

Theorem 5.8. *Let $\mu \geq 1$ and $\lambda \in [\mu, \infty)$ satisfy $\lambda^3 b^3 - \mu^3 a^3 = b^3 - a^3$. Let W satisfy*

$$W(\mathbf{F}) = \Phi(|\mathbf{F}|^2, |\operatorname{adj}\mathbf{F}|),$$

where $\Phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is convex and Φ is an increasing function in each of its arguments. Then $\mathbf{u}_\mu^{\mathbf{rad}}$, given by (2.3)–(2.4)*, is a global minimiser of the energy in $\mathcal{S}_\mu^p$ and also in $\widehat{\mathcal{S}}_\mu^p$.*

Remark 5.9. Suppose that, in addition to the hypotheses of the last theorem, there are constants $p > 3$, $c_0 > 0$, and $c_2 \geq 0$ such that, for all $s > 0$ and $t > 0$,

$$\Phi(s, t) \geq c_0 s^{\frac{p}{2}} - c_2.$$

Then one may then apply the theory of Ball [1] to deduce the existence of an absolute minimizer, $\mathbf{u}_m$, of the energy. Theorem 5.8 then shows that the radial incompressible minimiser, $\mathbf{u}_\mu^{\mathbf{rad}}$, is either equal to $\mathbf{u}_m$ or has the same energy as $\mathbf{u}_m$.

Proof of Theorem 5.8. We first note that since Φ is convex it is continuous; thus since $\mathbf{u}_\mu^{\mathbf{rad}} \in C^1(\overline{A}; \mathbb{R}^3)$ it follows that $\mathbf{u}_\mu^{\mathbf{rad}}$ has finite energy. Now let $\mathbf{u} \in \widehat{\mathcal{S}}_\mu^p$ (or $\mathcal{S}_\mu^p$). We will show that $\mathbf{u}_\mu^{\mathbf{rad}}$ has less energy than $\mathbf{u}$.

If $\mathbf{u}$ has infinite energy then the result is trivial since W is bounded below. Otherwise, since $\mathbf{u} \in \widehat{\mathcal{S}}_\mu^p$ and has finite energy, each of the integrals

$$\int_{S_R} \Phi(|\nabla\mathbf{u}|^2, |\operatorname{adj}\nabla\mathbf{u}|)\, d\mathcal{H}^2_{\mathbf{x}}, \quad \int_{S_R} |\nabla\mathbf{u}|^2\, d\mathcal{H}^2_{\mathbf{x}}, \quad \int_{S_R} |\operatorname{adj}\nabla\mathbf{u}|\, d\mathcal{H}^2_{\mathbf{x}}$$

must be finite for $\mathcal{L}^1$ almost every $R \in (a, b)$. Moreover, since $\det\nabla\mathbf{u} = 1$ a.e. it follows that, for $\mathcal{L}^1$ almost every $R \in (a, b)$,

$$\det\nabla\mathbf{u}(\mathbf{x}) = 1 \quad \text{for} \quad \mathcal{H}^2 \text{ a.e. } \mathbf{x} \in S_R$$

and hence, for such R, the arguments in section 3 (see Lemma 3.4, Lemma 3.12, and their proofs) yield, for $\mathcal{H}^2$ a.e. $\mathbf{x} \in S_R$,

$$g\left(\left|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right|\right) = 2\left|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right| + \left|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right|^{-2} \leq |\nabla\mathbf{u}|^2,$$

$$h\left(\left|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right|\right) = \sqrt{\left|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right|^2 + 2\left|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right|^{-1}} \leq |\operatorname{adj}\nabla\mathbf{u}|.$$

Finally, since $|\nabla\mathbf{u}|^2$, $|\operatorname{adj}\nabla\mathbf{u}|$, and $\Phi(|\nabla\mathbf{u}|^2, |\operatorname{adj}\nabla\mathbf{u}|)$ are $\mathcal{H}^2$ integrable for each such R the result follows from Proposition 5.5(b) and the corresponding proofs in section 3, with Lemma 5.7 replacing Proposition 3.2. □

We now give the analogues of the results in Section 4.

Theorem 5.10. *Let $\mu \in (0,1]$ and $\lambda > 0$ satisfy $\lambda^3 b^3 - \mu^3 a^3 = b^3 - a^3$. Let W satisfy*

$$W(\mathbf{F}) = \Psi(|\mathbf{F}|, |\operatorname{adj}\mathbf{F}|^2),$$

where $\Psi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is convex and Ψ is an increasing function in each of its arguments. Suppose that there are constants $p > 3$, $q > 3$, $c_0 > 0$, $c_1 > 0$, and $c_2 \geq 0$ such that, for all $s > 0$ and $t > 0$,

$$\Psi(s,t) \geq c_0 s^p + c_1 t^{\frac{q}{2}} - c_2. \tag{5.6}$$

Then $\mathbf{u}_\mu^{\mathbf{rad}}$, given by (2.3)–(2.4), is a global minimiser of the energy in $\mathcal{S}_\mu^p$.

Proof. In view of the proof of the previous result and without loss of generality, let $\mathbf{u} \in \mathcal{S}_\mu^p$ have finite energy. Then (5.6) implies that $\operatorname{adj}\nabla\mathbf{u} \in \mathcal{L}^q(A)$, where $q > 3$. Therefore, we can apply Proposition 5.3(c)–(e) to conclude that $\mathbf{u}$ has an inverse $\mathbf{v} \in W^{1,q}(A^*) \cap C^0(\overline{A^*})$ that satisfies $\nabla_{\mathbf{y}}\mathbf{v}(\mathbf{y}) = [\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{v}(\mathbf{y}))]^{-1}$ for a.e. $\mathbf{y} \in A^*$. Consequently, since $\mathbf{v}$ is isochoric, the change of variables formula[12] of Marcus and Mizel [16] yields (cf. the proof of Proposition 4.3)

$$\int_A \Psi\left(|\nabla\mathbf{u}(\mathbf{x})|, |\operatorname{adj}\nabla\mathbf{u}(\mathbf{x})|^2\right) d\mathbf{x} = \int_{A^*} \Psi\left(|\operatorname{adj}\nabla\mathbf{v}(\mathbf{y})|, |\nabla\mathbf{v}(\mathbf{y})|^2\right) d\mathbf{y}.$$

The desired result will then follow as in the proof of Theorem 5.8. □

Finally, Theorems 5.8 and 5.10 together yield the following result.

Corollary 5.11. *Let*

$$W(\mathbf{F}) = \Psi(|\mathbf{F}|^2, |\operatorname{adj}\mathbf{F}|^2),$$

where $\Psi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is convex, and Ψ is an increasing function in each of its arguments. Suppose, in addition, there are constants $p > 3$, $q > 3$, $c_0 > 0$, $c_1 > 0$, and $c_2 \geq 0$ such that, for all $s > 0$ and $t > 0$,

$$\Phi(s,t) \geq c_0 s^{\frac{p}{2}} + c_1 t^{\frac{q}{2}} - c_2.$$

Then, for every $\mu > 0$, $\mathbf{u}_\mu^{\mathbf{rad}}$ given by (2.3)–(2.4) is a global minimiser of the energy in $\mathcal{S}_\mu^p$.

6. Concluding Remarks.

In this section we indicate some of the implications and possible extensions of the methods and results developed in this current paper to other problems in nonlinear elasticity.

Compressible Materials. The results on symmetry obtained in this paper can be further developed to apply to compressible materials to prove that that the corresponding radially symmetric compressible equilibria are energy minimisers (see [22]).

[12] See also, e.g., [9, p. 141], [24, Theorem 2], and [8, Theorems 2.10.43, 3.1.8, and 3.2.5].

Cavitation. The methods in this paper can also be adapted to study discontinuous equilibria arising in the study of cavitation of a solid incompressible elastic ball (see, e.g, [4]) to prove that the radially symmetric solution (given by (1.9) on the entire ball $B_b = \{\mathbf{x} : |\mathbf{x}| < b\}$) is a global energy minimiser amongst all deformations opening a cavity at the centre of the ball (see [23]).

Pressure Loading.

Another problem of interest for an annular region is the existence and stability of equilibrium solutions when such a thick spherical shell is subject to a pressure on its inner boundary. For example, Haughton and Ogden [11] (see, also, [14]) determine constitutive restrictions that are necessary and also others that are sufficient for the radial minimizer to become linearization unstable.[13] One might then expect a second solution branch would bifurcate from the radial solution branch and, when this second branch is supercritical, the solutions on it should have less energy than the radial solution. We now show that our constitutive restrictions complement those of [11] in the sense that our hypotheses imply that the radial minimizer is a strong relative minimizer for this problem when the pressure is positive.

Definition 6.1. For the pressure problem we fix a constant pressure $P > 0$ and define the set of admissible deformations and the total energy, respectively, by

$$\mathcal{A} := \left\{\mathbf{u} \in C^1(\overline{A}; \mathbb{R}^3) : \mathbf{u} \text{ is one-to-one and } \det \nabla\mathbf{u} \equiv 1\right\},$$

$$\mathbb{E}_P(\mathbf{u}) := \int_A W(\nabla\mathbf{u}(\mathbf{x}))\, d\mathbf{x} - P \int_{\mathbf{u}(S_a)} \mathbf{y}\cdot\mathbf{N}(\mathbf{y})\, dS_{\mathbf{y}}$$

for any $\mathbf{u} \in \mathcal{A}$, where $\mathbf{N}$ is the outward unit normal to $\mathbf{u}(A)$.

Let's now restrict our attention to stored energies W of the form

$$W(\mathbf{F}) = \Phi(|\mathbf{F}|^2, |\operatorname{adj}\mathbf{F}|),$$

where $\Phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is convex and an increasing function in each of its arguments. Also, suppose there are constants $q > 3$, $c_0 > 0$, and $c_2 \geq 0$ such that, for all $s > 0$ and $t > 0$,

$$\Phi(s, t) \geq c_0 s^{\frac{q}{2}} - c_2. \tag{6.1}$$

In particular for the radial deformation given by (2.3)–(2.4) the total energy is equal to

$$\widehat{\mathbb{E}}_P(\tau) = E^{\text{elas}}(\tau) - 4\pi P a^3 \tau, \qquad E^{\text{elas}}(\tau) = 4\pi \int_a^b \widehat{\Phi}(R, \tau) R^2\, dR,$$

$$\widehat{\Phi}(R, \tau) := \Phi\Big(\mathbb{D}\big(r, R\big), \mathbb{A}\big(r, R\big)\Big),$$

[13] That is, the linear elliptic system of partial differential equations, which one obtains upon linearizing the equilibrium equations about the radial equilibrium solution, has a nontrivial solution.

where $\tau = \mu^3$, $r^3 = r(R,\tau)^3 = R^3 + a^3(\tau - 1)$,

$$\mathbb{D}(r,R) = \left(\frac{R}{r}\right)^4 + 2\left(\frac{r}{R}\right)^2, \quad \mathbb{A}(r,R) = \left[\left(\frac{r}{R}\right)^4 + 2\left(\frac{R}{r}\right)^2\right]^{\frac{1}{2}}.$$

We note that $\mathbb{D}(r,R) > 2b^{-2}r^2 \geq 2b^{-2}r(a,\tau)^2 = 2\tau^{2/3}(a/b)^2$ and hence (6.1) implies

$$\widehat{\mathbb{E}}_P(\tau) \geq C_1\tau^{q/3} - C_2P\tau$$

for some positive constants C_1 and C_2. Therefore, $q > 3$ yields $\widehat{\mathbb{E}}_P \to +\infty$ as $\tau \to +\infty$. Also,

$$\frac{\partial}{\partial r}\mathbb{D}(r,R) = \frac{4}{r^5R^2}\left[r^6 - R^6\right], \quad \frac{\partial}{\partial r}\mathbb{A}(r,R) = \frac{2}{r^3R^4\mathbb{A}}\left[r^6 - R^6\right], \quad \frac{\partial}{\partial \tau}r(R,\tau) = \frac{a^3}{3r^2}. \quad (6.2)$$

Now, $r(R,\tau) < R$ when $\tau < 1$ and hence, in view of (6.2) and the monotonicity of Φ, $\frac{d}{d\tau}E^{\text{elas}}(\tau) < 0$ for $\tau < 1$. Thus, for P strictly positive, $\frac{d}{d\tau}\widehat{\mathbb{E}}_P(\tau) < 0$ on $(0,1]$. We have therefore shown that *for each pressure $P > 0$ there is at least one minimizer of the total energy $\mathbb{E}_P$ amongst all isochoric radial deformations r_μ^{inc}, $\mu \in \mathbb{R}^+$. Moreover, this minimizer satisfies $\mu > 1$ and hence $r_\mu^{\text{inc}}(a) = a\mu > a$.*

Next, for each $\mathbf{u} \in \mathcal{A}$ choose $\sigma \in \mathbb{R}^+$ so that $\mathbf{u}(S_a)$ and $\mathbf{u}_\sigma^{\mathbf{rad}}(S_a)$ enclose the same volume, that is,

$$[r_\sigma^{\text{inc}}(a)]^3 = a^3\sigma^3 = \frac{1}{4\pi}\int_{\mathbf{u}(S_a)} \mathbf{y}\cdot\mathbf{N}(\mathbf{y})\,dS_{\mathbf{y}}.$$

Then the energy contribution from the boundary load is the same for $\mathbf{u}$ and $\mathbf{u}_\sigma^{\mathbf{rad}}$; the results in Section 3 show that, whenever $\sigma \geq 1$, the elastic energy of $\mathbf{u}$ is greater than or equal to the elastic energy of $\mathbf{u}_\sigma^{\mathbf{rad}}$. We have thus established the following result.

Theorem 6.2. *Suppose that the previously stated convexity, monotonicity, and growth conditions on Φ are satisfied. Then for any $P > 0$ there exists a $\mu > 1$ such that $\mathbf{u}_\mu^{\mathbf{rad}}$ given by* (2.3)–(2.4) *is a strong relative minimizer of the energy, i.e., there is an $\varepsilon_P > 0$ such if $\mathbf{u} \in \mathcal{A}$ satisfies*

$$\sup_{\mathbf{x}\in\overline{A}} |\mathbf{u}_\mu^{\mathbf{rad}}(\mathbf{x}) - \mathbf{u}(\mathbf{x})| < \varepsilon_P \quad \textit{then} \quad \mathbb{E}(\mathbf{u}) \geq \mathbb{E}(\mathbf{u}_\mu^{\mathbf{rad}}).$$

Remark 6.3. Haughton and Ogden [11] show that for linearization instability to occur the constitutive relation *must* satisfy

$$B(\nu) := \nu\widehat{W}''(\nu) - \widehat{W}'(\nu) < 0 \quad \text{for some } \nu > 1,$$

where (in our notation)

$$\widehat{W}(\nu) := \Phi\left(\nu^{-4} + 2\nu^2, \left[\nu^4 + 2\nu^{-2}\right]^{1/2}\right), \qquad \widehat{W}' := \frac{d}{d\nu}\widehat{W}.$$

A straightforward computation shows that Φ convex and monotone increasing in each of its arguments yields $B(\nu) \geq 0$ for all $\nu > 1$, which prohibits bifurcation in tension.

Remark 6.4. It is clear from our proofs that if, in addition, the displacement is prescribed on the outer boundary, i.e., $\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x}$ for $\mathbf{x} \in S_b$, where $\lambda \geq 1$ then the radial minimizer is a global mininimiser of the total energy $\mathbb{E}_P$ for any $P \in \mathbb{R}$.

An example of Fritz John revisited.

Motivated by a heuristic example originally due to F. John, the paper [20] studies, in particular, energy minimising deformations $\mathbf{u} : A \to \mathbb{R}^2$ of a two-dimensional annulus $A = \{\mathbf{x} \in \mathbb{R}^2 : a < |\mathbf{x}| < b\}$, which equal the identity on ∂A. It is shown therein that there exist energy minimisers to this displacement boundary-value problem in various homotopy classes of deformations of A. Each homotopy class corresponds to deformations that fix one of the boundaries of A and twist the other boundary through an integer multiple $N \in \mathbb{Z}$ of 2π. Suppose that we now restrict attention to isochoric deformations and generalise the boundary condition from $\mathbf{u}(\mathbf{x}) = \mathbf{x}$ on ∂A as considered in [20] to $\mathbf{u}(\mathbf{x}) = \mu\mathbf{x}$ on ∂A_I and $\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x}$ on ∂A_o (as in (1.6)). Then the two-dimensional versions of the arguments contained in sections 3 and 5 can be adapted (see Remarks 3.10 and 3.17) to show that, for stored-energy functions of the form (1.4), the radially symmetric deformation of the annulus given by (2.3)–(2.4) is a global energy minimiser amongst all general isochoric deformations of the annulus. Although our results do not demonstrate that any of the twisted solutions has *strictly*[14] greater energy, an argument that shows that a solution with a sufficient number of twists has strictly greater energy than the radial minimiser is given in [20, Section 3].

As an example, consider deformations that have the special form (2.9) for a neo-Hookean material, $W(\mathbf{F}) = \frac{\mu}{2}|\mathbf{F}|^2$. Then an analysis similar to that in [20, Section 4] shows that there are an infinite number of *equilibrium solutions*, one for each winding number $N \in \mathbb{Z}$. The function ρ for all such solutions is given by (2.11). A straightforward calculation yields the corresponding equilibrium "twist" functions (recall $\psi(b) = 2N\pi$)

$$\psi(R) = \frac{K}{(\lambda^2-1)b^2}\left[\ln\left(\frac{R}{a}\right) - \frac{1}{2}\ln\left(\frac{R^2+(\lambda^2-1)b^2}{a^2+(\lambda^2-1)b^2}\right)\right],$$

where the constant K is determined by the number of twists and is given by

$$K = \frac{4N\pi(\lambda^2-1)b^2}{\ln\left(\frac{a^2+(\lambda^2-1)b^2}{a^2\lambda^2}\right)}.$$

Our results show that the radial function, which corresponds to $K = N = 0$ and $\psi \equiv 0$, is an energy minimiser; however, one can also obtain this result directly from (2.10).

[14] However, see the appendix of this paper.

A. Appendix

We here address the problem of whether or not the radial minimizer is a *strict* global minimizer of the energy. We will assume that $W(\mathbf{F}) = \Phi(|\mathbf{F}|^2, |\operatorname{adj}\mathbf{F}|)$, where Φ is *strictly convex*, i.e., if $z \neq a$ or $w \neq b$

$$\Phi(z,w) > \Phi(a,b) + \Phi_{,1}(a,b)(z-a) + \Phi_{,2}(a,b)(w-b).$$

Before proceeding with our analysis, we first note (cf. Proposition 3.2) that the isoperimetric inequality is a *strict inequality* unless the image of each spherical shell centred at the origin is again a spherical shell.

We will prove the following result.

Proposition A.1. *Let $W(\mathbf{F}) = \Phi(|\mathbf{F}|^2, |\operatorname{adj}\mathbf{F}|)$, where Φ is strictly convex and an increasing function in each of its arguments. Let $\mu > 0$ and suppose that $\mathbf{u} \in \mathcal{A}^I_\mu$ satisfies $E(\mathbf{u}) = E(\mathbf{u}^{\mathbf{rad}}_\mu)$. Then $\mathbf{u} \equiv \mathbf{u}^{\mathbf{rad}}_\mu$ is radial.*

This result will follow immediately from the three Lemma's below.

Lemma A.2. *Let $W(\mathbf{F}) = \Phi(|\mathbf{F}|^2, |\operatorname{adj}\mathbf{F}|)$, where Φ is strictly convex and an increasing function in each of its arguments. Let $\mu > 0$ and suppose that $\mathbf{u} \in \mathcal{A}^I_\mu$ satisfies $E(\mathbf{u}) = E(\mathbf{u}^{\mathbf{rad}}_\mu)$. Then each of the eigenvalues of the right Cauchy-Green strain tensor $\mathbf{C}(\mathbf{x}) := [\nabla\mathbf{u}(\mathbf{x})]^{\mathrm{T}}\nabla\mathbf{u}(\mathbf{x})$ is radial and $\mathbf{x}$ is an eigenvector of $\mathbf{C}(\mathbf{x})$.*

Lemma A.3. *Let $\mu > 0$ and $\mathbf{u} \in \mathcal{A}^I_\mu$. Suppose that the image of each spherical shell centred at the origin is again a spherical shell. More precisely suppose that, for each $R \in [a,b]$, there exists a point $\mathbf{z}(R) \in \mathbb{R}^3$ and a scalar $\beta(R) \geq 0$ such that*

$$|\mathbf{u}(\mathbf{x}) - \mathbf{z}(R)|^2 = 2\beta(R), \quad R := |\mathbf{x}| \quad \textit{for every} \quad \mathbf{x} \in S_R. \tag{A.1}$$

Then $\mathbf{z} \in C^1([a,b];\mathbb{R}^3)$ and $\beta \in C^1([a,b];\mathbb{R})$ with $\beta > 0$ and $\beta' > 0$.

Lemma A.4. *Let $\mu > 0$ and suppose that $\mathbf{u} \in \mathcal{A}^I_\mu$ satisfies* (A.1)*, where $\mathbf{z} \in C^1([a,b];\mathbb{R}^3)$ and $\beta \in C^1([a,b];\mathbb{R})$ with $\beta > 0$. Suppose in addition that, for each $\mathbf{x} \in \overline{A}$, $\mathbf{x}$ is an eigenvector of the right Cauchy-Green strain tensor $\mathbf{C}(\mathbf{x}) := [\nabla\mathbf{u}(\mathbf{x})]^{\mathrm{T}}\nabla\mathbf{u}(\mathbf{x})$ and that the corresponding eigenvalue $\sigma = \sigma(|\mathbf{x}|)$ is radial, i.e.,*

$$\big([\nabla\mathbf{u}(\mathbf{x})]^{\mathrm{T}}\nabla\mathbf{u}(\mathbf{x})\big)\mathbf{x} = \sigma(R)\mathbf{x}. \tag{A.2}$$

Then $\mathbf{u} \equiv \mathbf{u}^{\mathbf{rad}}_\mu$ is radial.

Proof of Lemma A.2. Let $\mu > 0$ and $\mathbf{u} \in \mathcal{A}^I_\mu$. Then in view of Remark 3.16 the energy of $\mathbf{u}$ will be strictly greater than the energy of $\mathbf{u}^{\mathbf{rad}}_\mu$ unless $\mathbf{x}$ is an eigenvector of $\mathbf{C}(\mathbf{x})$. Next, for each $\mathbf{x} \in \overline{A}$ let $\lambda_i(\mathbf{x})$ denote the eigenvalues of $\mathbf{C}(\mathbf{x})$. It then follows from the proof of

Theorem 3.15 and the hypothesis that Φ is strictly convex that the energy of $\mathbf{u}$ is strictly greater that the energy of $\mathbf{u}_\mu^{\mathbf{rad}}$ unless $|\nabla \mathbf{u}|^2$ and $|\operatorname{adj} \nabla \mathbf{u}|$ are both radial. Thus we find with the aid of the identity $\operatorname{tr}(\operatorname{adj} \mathbf{C}) = \frac{1}{2}[(\operatorname{tr}(\mathbf{C}))^2 - \operatorname{tr}(\mathbf{C}^2)]$ that

$$\begin{aligned} \tau(R) &:= |\nabla \mathbf{u}|^2 = \operatorname{tr}(\mathbf{C}) = \lambda_1(\mathbf{x}) + \lambda_2(\mathbf{x}) + \lambda_3(\mathbf{x}), \\ \alpha(R) &:= |\operatorname{adj} \nabla \mathbf{u}|^2 = \operatorname{tr}(\operatorname{adj} \mathbf{C}) = \lambda_2(\mathbf{x})\lambda_3(\mathbf{x}) + \lambda_1(\mathbf{x})\lambda_3(\mathbf{x}) + \lambda_1(\mathbf{x})\lambda_2(\mathbf{x}), \\ 1 &= (\det \nabla \mathbf{u})^2 = \det \mathbf{C} = \lambda_1(\mathbf{x})\lambda_2(\mathbf{x})\lambda_3(\mathbf{x}). \end{aligned} \tag{A.3}$$

However, the eigenvalues of $\mathbf{C}$ satisfy the characteristic equation

$$\lambda^3 - \tau(R)\lambda^2 + \alpha(R)\lambda - 1 = 0,$$

which implies that the eigenvalues depend only on R. $\square$

Remark A.5. If instead $\mathbf{u} \in \mathcal{S}_\mu^p$ (or $\widehat{\mathcal{S}}_\mu^p$) then, for $\mathcal{L}^1$ a.e. $R \in (a,b)$, $(\text{A.3})_{1-3}$ are satisfied for $\mathcal{H}^2$ a.e. $\mathbf{x} \in S_R$. Thus the fact that the eigenvalues satisfy the characteristic equation again yields their dependence on R alone.

Proof of Lemma A.3. Let $\mu > 0$, $\mathbf{u} \in \mathcal{A}_\mu^I$, and fix $R \in [a,b]$. Then (3.7) (the change of variables formula) states

$$\frac{1}{3}\int_{S_R} \mathbf{u} \cdot (\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}} \mathbf{n} = \frac{1}{3}\int_{\mathbf{u}(S_R)} \mathbf{y} \cdot \mathbf{N}, \tag{A.4}$$

where $\mathbf{n}$ is the outward unit normal to B_R and $\mathbf{N}$ is the outward unit normal to the ball defined by (A.1). However, by the divergence theorem, the integral on the right-hand side of (A.4) is equal to the volume of this ball. Also, (3.5) and (3.6) imply that the integral on the left-hand side of (A.4) is equal to the volume of the ball of radius $r_\mu^{\mathrm{inc}}(R)$. Thus, in view of (2.4) we conclude that

$$\beta(R)^3 = \frac{1}{8}\left[R^3 + a^3(\mu^3 - 1)\right]^2. \tag{A.5}$$

Therefore, $\beta \in C^1([a,b])$, $\beta > 0$, and $\beta' > 0$, as claimed.

Next, replace $\mathbf{y}$ in the above computation by the tensor $\frac{3}{4}\mathbf{y} \otimes \mathbf{y}$, i.e., use the change of variables formula for surface integrals with the tensor $\mathbf{y} \otimes \mathbf{y} = \mathbf{u}(\mathbf{x}) \otimes \mathbf{u}(\mathbf{x})$. Then (A.4) is replaced by

$$\frac{1}{4}\int_{S_R} [\mathbf{u} \otimes \mathbf{u}] (\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}} \mathbf{n} = \frac{1}{4}\int_{\mathbf{u}(S_R)} [\mathbf{y} \otimes \mathbf{y}]\, \mathbf{N}. \tag{A.6}$$

Again the divergence theorem, together with a straightforward computation, shows that the integral on the right-hand side of (A.6) is equal to the volume of this ball times $\mathbf{z}(R)$. (Here we use the fact that the integral of $\mathbf{y}$ over this ball is equal to the integral of the constant vector $\mathbf{z}(R)$ over this ball since $\mathbf{y} = (\mathbf{y} - \mathbf{z}) + \mathbf{z}$ and $\mathbf{y} - \mathbf{z}$ is normal to the ball.)

Next, we consider the left-hand side of (A.6). Suppose that $\mathbf{w} \in C^2(\overline{A};\mathbb{R}^3)$. Then, in view of the identity $\operatorname{div}(\operatorname{adj}\nabla\mathbf{w})^{\mathrm{T}} = \mathbf{0}$,

$$[\mathbf{w}\otimes\mathbf{w}]\,(\operatorname{adj}\nabla\mathbf{w})^{\mathrm{T}} = \mathbf{w}\otimes[(\operatorname{adj}\nabla\mathbf{w})\mathbf{w}], \qquad (\operatorname{adj}\nabla\mathbf{w})\nabla\mathbf{w} = (\det\nabla\mathbf{w})\mathbf{I},$$

$$\operatorname{div}(\mathbf{w}\otimes\mathbf{v}) = (\nabla\mathbf{w})\mathbf{v} + \mathbf{w}\operatorname{div}\mathbf{v}, \qquad \operatorname{div}\left((\operatorname{adj}\nabla\mathbf{w})\mathbf{w}\right) = (\operatorname{adj}\nabla\mathbf{w})^{\mathrm{T}} : \nabla\mathbf{w},$$

and hence

$$\operatorname{div}\left([\mathbf{w}\otimes\mathbf{w}]\,(\operatorname{adj}\nabla\mathbf{w})^{\mathrm{T}}\right) = 4(\det\nabla\mathbf{w})\mathbf{w}. \tag{A.7}$$

If we then integrate (A.7) over the region $B_R\backslash B_a$ and make use of the divergence theorem we find that

$$\int_{S_R}[\mathbf{w}\otimes\mathbf{w}]\,(\operatorname{adj}\nabla\mathbf{w})^{\mathrm{T}}\mathbf{n} = \int_{S_a}[\mathbf{w}\otimes\mathbf{w}]\,(\operatorname{adj}\nabla\mathbf{w})^{\mathrm{T}}\mathbf{n} + 4\int_{B_R\backslash B_a}(\det\nabla\mathbf{w})\mathbf{w}\,d\mathbf{x}. \tag{A.8}$$

However, just as in the proof of Proposition 3.2, we now note that C^2 is dense in C^1 and so the bounded convergence theorem yields (A.8) for all $\mathbf{w}\in C^1(\overline{A};\mathbb{R}^3)$ and, in particular, for all $\mathbf{w}\in\mathcal{A}^I_\mu$.

We now consider the first integral on the right-hand side of (A.8) with $\mathbf{w}=\mathbf{u}$. The boundary condition $\mathbf{u}(\mathbf{x}) = \mu\mathbf{x}$ for $\mathbf{x}\in S_a$ implies $[\mathbf{u}\otimes\mathbf{u}]\,(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n} = \mu^4a^2\mathbf{n}$ and so an application of the divergence theorem over the ball B_a yields

$$\int_{S_a}[\mathbf{u}\otimes\mathbf{u}]\,(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n} = \mathbf{0}. \tag{A.9}$$

Finally, we combine (A.8)–(A.9) to conclude, with the aid of the (A.6) and the text immediately following it, that $\mathbf{z}$ times the volume of the ball given by (A.1) is equal to the final integral on the right-hand side of (A.8) with $\mathbf{w}=\mathbf{u}$ and $\det\nabla\mathbf{u} = 1$, that is,

$$\frac{4}{3}\pi\left(\sqrt{2\beta(R)}\right)^3\mathbf{z}(R) = \int_{B_R\backslash B_a}\mathbf{u}(\mathbf{x})\,d\mathbf{x}. \tag{A.10}$$

The continuity of $\mathbf{u}$ together with the strict positivity of β and β' then yield $\mathbf{z}\in C^1([a,b];\mathbb{R}^3)$. $\square$

Remark A.6. If one replaces the classical area formulae (A.4) and (A.6) with their appropriate Sobolev version, i.e., Proposition 5.5(e,f) with $\mathbf{v}(\mathbf{y}) = \mathbf{y}/3$ and $\mathbf{V}(\mathbf{y}) = \frac{3}{4}\mathbf{y}\otimes\mathbf{y}$, then it is clear from the above proof (and the proof of Lemma 5.7) that (A.5) and (A.10) are also satisfied, for a.e. $R\in(a,b)$, when $\mathbf{u}\in\mathcal{S}^p_\mu$ (or $\widehat{\mathcal{S}}^p_\mu$). Consequently, Lemma A.3 is also valid for such $\mathbf{u}$.

Proof of Lemma A.4. Let $\mu>0$ and $\mathbf{u}\in\mathcal{A}^I_\mu$. Define

$$\mathbf{v}(\mathbf{x}) := \mathbf{u}(\mathbf{x}) - \mathbf{z}(R), \qquad \nabla\mathbf{v} = \nabla\mathbf{u} - \mathbf{z}'\otimes\mathbf{n}, \qquad \mathbf{n} := \frac{\mathbf{x}}{|\mathbf{x}|}, \qquad R := |\mathbf{x}|. \tag{A.11}$$

Then $|\mathbf{v}|^2 = 2\beta(R)$ and hence if we take the gradient with respect to $\mathbf{x}$ we find that

$$\beta'\mathbf{n} = (\nabla\mathbf{v})^{\mathrm{T}}\mathbf{v} = (\nabla\mathbf{u})^{\mathrm{T}}\mathbf{v} - \left[\mathbf{n}\otimes\mathbf{z}'\right]\mathbf{v}. \tag{A.12}$$

Next, multiply (A.12) by $(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}$ and rearrange terms to conclude

$$\left[\beta' + \mathbf{z}'\cdot\mathbf{v}\right](\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n} = \mathbf{v}.$$

We now apply hypothesis (A.2); equivalently (cf. Remark 3.16),

$$\alpha(R)(\nabla\mathbf{u})\mathbf{n} = (\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}, \qquad \alpha := 1/\sigma,$$

which when combined with the previous equation and $(\mathrm{A.11})_2$ implies

$$\alpha\left[\beta' + \mathbf{z}'\cdot\mathbf{v}\right]\left[(\nabla\mathbf{v})\mathbf{n} + \mathbf{z}'\right] = \mathbf{v}. \tag{A.13}$$

Also, the first equality in (A.12) yields

$$\beta' = \mathbf{n}\cdot\left(\beta'\mathbf{n}\right) = \mathbf{n}\cdot(\nabla\mathbf{v})^{\mathrm{T}}\mathbf{v} = (\nabla\mathbf{v})\mathbf{n}\cdot\mathbf{v}. \tag{A.14}$$

Thus, if we take the inner product of (A.13) with $\mathbf{v}$ and make use of (A.14) and the fact that $\mathbf{v}\cdot\mathbf{v} = 2\beta$ we find that

$$\beta' + \mathbf{z}'\cdot\mathbf{v} = \sqrt{\frac{2\beta}{\alpha}}. \tag{A.15}$$

We next substitute (A.15) into (A.13) to get

$$\sqrt{2\alpha\beta}\left[\mathbf{v}_R + \mathbf{z}'\right] = \mathbf{v}, \tag{A.16}$$

where $\mathbf{v}_R := (\nabla\mathbf{v})\mathbf{n}$. For each fixed unit vector $\mathbf{n} = \mathbf{x}/|\mathbf{x}|$ we view (A.16) as a system of ordinary differential equations for the function $R \mapsto \mathbf{v}(R,\mathbf{n})$. The general solution of (A.16) is $\mathbf{v}(R,\mathbf{n}) = \widehat{\mathbf{q}}(R) + \delta(R)\mathbf{w}(\mathbf{n})$ for appropriate functions $\widehat{\mathbf{q}} \in C^1([a,b];\mathbb{R}^3)$, $\delta \in C^1([a,b];\mathbb{R})$, and $\mathbf{w} \in C^1(S^2;\mathbb{R}^3)$. Consequently, in view of $(\mathrm{A.11})_1$,

$$\mathbf{u}(\mathbf{x}) = \mathbf{q}(R) + \delta(R)\mathbf{w}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right). \tag{A.17}$$

We next note that since $\mathbf{u} \in \mathcal{A}^I_\mu$ it follows that $\mathbf{u}(\mathbf{x}) = \mu\mathbf{x}$ for $\mathbf{x} \in S_a$. Thus by (A.17)

$$\mu|\mathbf{x}|\frac{\mathbf{x}}{|\mathbf{x}|} = \mathbf{u}(\mathbf{x}) = \mathbf{q}(a) + \delta(a)\mathbf{w}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \quad \text{for all } \mathbf{x} \in S_a.$$

Therefore, there are constants $\kappa \in \mathbb{R}$ and $\mathbf{u}_0 \in \mathbb{R}^3$ such that

$$\mathbf{w}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) = \kappa\frac{\mathbf{x}}{|\mathbf{x}|} + \mathbf{u}_0 \quad \text{for all } \mathbf{x} \in A,$$

which together with (A.17) yields

$$\mathbf{u}(\mathbf{x}) = r(R)\frac{\mathbf{x}}{|\mathbf{x}|} + \mathbf{c}(R) \quad \text{for all } \mathbf{x} \in A. \tag{A.18}$$

Next, we take the gradient of (A.18) with respect to $\mathbf{x}$ and find that

$$\nabla\mathbf{u}(\mathbf{x}) = \frac{r(R)}{R}\mathbf{I} + \left(r' - \frac{r(R)}{R}\right)\frac{\mathbf{x}}{R} \otimes \frac{\mathbf{x}}{R} + \mathbf{c}' \otimes \mathbf{n}.$$

Consequently, $[\nabla\mathbf{u}]\mathbf{n} = r'\mathbf{n} + \mathbf{c}'$ and hence

$$\left[(\nabla\mathbf{u})^{\mathrm{T}}\,\nabla\mathbf{u}\right]\mathbf{n} = \eta(R)\mathbf{n} + \frac{r(R)}{R}\mathbf{c}'.$$

Now recall that $\mathbf{n}$ is an eigenvector of $(\nabla\mathbf{u})^{\mathrm{T}}\,\nabla\mathbf{u}$. Therefore, since R and r are nonzero,

$$\mathbf{c}'(R) = |\mathbf{c}'(R)|\frac{\mathbf{x}}{|\mathbf{x}|},$$

which is only possible if $\mathbf{c}' \equiv 0$. It follows that $\mathbf{c}(R) \equiv \mathbf{c}_0$, a constant, and so (A.18) reduces to

$$\mathbf{u}(\mathbf{x}) = r(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|} + \mathbf{c}_0 \quad \text{for all } \mathbf{x} \in A. \tag{A.19}$$

Finally, apply the boundary condition $\mathbf{u}(\mathbf{x}) = \mu\mathbf{x}$ for $\mathbf{x} \in S_a$ once again, this time to (A.19), to conclude that, for such $\mathbf{x}$,

$$\mu\mathbf{x} = \mathbf{u}(\mathbf{x}) = r(a)\frac{\mathbf{x}}{|\mathbf{x}|} + \mathbf{c}_0,$$

or, equivalently,

$$\mathbf{c}_0 = (\mu a - r(a))\,\frac{\mathbf{x}}{|\mathbf{x}|}.$$

Therefore, $r(a) = \mu a$, $\mathbf{c}_0 = \mathbf{0}$, and consequently $\mathbf{u}$ is radial, as claimed. □

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