On the Symmetry of Energy Minimising Deformations in Nonlinear Elasticity II: Compressible Materials

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Abstract. Consider a homogeneous, isotropic, hyperelastic body occupying the annular region $A = \{\mathbf{x} \in \mathbb{R}^n : a < |\mathbf{x}| < b\}$ in its reference state and subject to radially symmetric displacement, or mixed displacement/traction, boundary conditions. In Part I [18] it was shown that if the body is composed of an incompressible material, then to each isochoric deformation of A there corresponds a radial isochoric deformation that has less elastic energy than the given deformation, provided that the stored-energy function is polyconvex and grows sufficiently rapidly at infinity. In this paper that analysis is further developed and extended to the compressible case for a large class of polyconvex constitutive relations.

The key ingredient is a new radial-symmetrisation procedure that is appropriate for problems where the symmetrised mapping must be one-to-one in order to prevent interpenetration of matter. For the pure displacement boundary-value problem, the radial symmetrisation of an orientation preserving diffeomorphism $\mathbf{u} : A \to A^*$ between annuli A and A^* is the deformation

$$\mathbf{u}^{\mathrm{rad}}(\mathbf{x}) = \frac{r(R)}{R}\mathbf{x}, \qquad R = |\mathbf{x}|,$$

that maps each sphere $S_R \subset A$, of radius $R > 0$, centred at the origin into another such sphere $S_r = \mathbf{u}^{\mathrm{rad}}(S_R) \subset A^*$ that encloses the same volume as $\mathbf{u}(S_R)$. Since the volumes enclosed by the surfaces $\mathbf{u}(S_R)$ and $\mathbf{u}^{\mathrm{rad}}(S_R)$ are equal, the classical isoperimetric inequality then implies that $\mathrm{Area}(\mathbf{u}^{\mathrm{rad}}(S_R)) \le \mathrm{Area}(\mathbf{u}(S_R))$. The equality of the enclosed volumes together with this reduction in surface area is then shown to give rise to a reduction in total energy for many of the constitutive relations used in nonlinear elasticity.

These results are also extended to classes of Sobolev deformations and applied to prove that the radially symmetric solutions to these boundary-value problems are local or global energy minimisers in various classes of (possibly non-symmetric) deformations of the annulus.

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1 Introduction

Let $A = \{\mathbf{x} \in \mathbb{R}^n : a < |\mathbf{x}| < b\}$ (with $n = 2$ or $n = 3$) be an annulus centred at the origin. Consider a compressible, hyperelastic material that occupies the region A in its reference configuration and let $\mathbf{u} \in C^1(\overline{A}; \mathbb{R}^n)$ (or more generally $\mathbf{u}$ in an appropriate Sobolev space) be a one-to-one mapping with strictly positive Jacobian. Then to each such deformation we associate a corresponding energy

$$E(\mathbf{u}) = \int_A W(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x}, \tag{1.1}$$

where $W : \mathrm{M}^{n\times n}_+ \to [0, \infty)$ is the stored-energy function and $\mathrm{M}^{n\times n}_+$ denotes the set of $n \times n$ matrices with (strictly) positive determinant. We assume that the stored-energy function is both isotropic and frame indifferent:

$$W(\mathbf{FQ}) = W(\mathbf{QF}) = W(\mathbf{F}) \ \text{ for all } \mathbf{F} \in \mathrm{M}^{n\times n}_+ \text{ and } \mathbf{Q} \in SO(n),$$

where $SO(n)$ denotes the special orthogonal group of $n \times n$ matrices.

We further assume that, when $n = 2$, the stored-energy function is of the form

$$W(\mathbf{F}) = \Phi(|\mathbf{F}|^2, \det \mathbf{F}), \tag{1.2}$$

where $\Phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is convex and $s \mapsto \Phi(s, j)$ is monotone increasing for all $j > 0$, while, for $n = 3$,

$$W(\mathbf{F}) = \Phi(|\mathbf{F}|^3, |\operatorname{adj} \mathbf{F}|^{\frac{3}{2}}, \det \mathbf{F}), \tag{1.3}$$

where $\Phi : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is convex and $s \mapsto \Phi(s, t, j)$ and $s \mapsto \Phi(t, s, j)$ are monotone increasing for all $t > 0$ and $j > 0$. Thus each such energy function is homogeneous, isotropic, and polyconvex.

Fix $\lambda > 0$ and $\mu > 0$ with $\mu a < \lambda b$. The main boundary-value problem we consider is the **pure displacement problem** where we specify

$$\mathbf{u}(\mathbf{x}) = \mu \mathbf{x} \ \text{ for } \ |\mathbf{x}| = a, \qquad \mathbf{u}(\mathbf{x}) = \lambda \mathbf{x} \ \text{ for } \ |\mathbf{x}| = b.$$

We also consider the **mixed problem** where the displacement is only specified on one surface of the annulus while the other surface is left free of traction.

For polyconvex stored-energy functions W, the existence theory of Ball [1] and subsequent generalisations give hypotheses under which a minimiser of (1.1) exists for the above boundary-value problems. In parallel to this work, there are results on the existence and properties of minimisers in the class of radial (rotationally symmetric) deformations (see, e.g., [4, 17]).

Our main result is that for the displacement problem for the annulus and for stored-energy functions of the form (1.2) and (1.3) with[1] $\mu \geq \lambda$, the global energy minimiser must be radially symmetric (see Theorems 5.12 and 8.1 and Remark 5.15). The techniques and results obtained also have implications for mixed displacement/traction problems. The main ideas used in this paper originate in our prior work for incompressible materials [18], which we will henceforth refer to as Part I.

Our results are most easily described by focussing on one problem, for example, the case $n = 3$ for the mixed displacement/traction problem given above in which we specify the displacement on the outer boundary and the inner boundary is left free. For this problem let $\mathbf{u} \in C^1(\overline{A}; \mathbb{R}^3)$ be a deformation of the annulus satisfying the boundary condition $\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x}$ for $|\mathbf{x}| = b$. We define the *radial symmetrisation* $\mathbf{u}^{\mathrm{rad}}$ of $\mathbf{u}$ to be the radial deformation

$$\begin{aligned} \mathbf{u}^{\mathrm{rad}}(\mathbf{x}) &= \frac{r(R)}{R}\mathbf{x}, \qquad R := |\mathbf{x}|, \\ \frac{4}{3}\pi r(R)^3 &:= \frac{4}{3}\pi\lambda^3 b^3 - \int_{B_b \setminus B_R} (\det \nabla\mathbf{u})\, d\mathbf{x}, \end{aligned} \tag{1.4}$$

where B_R is the ball of radius R centred at the origin.

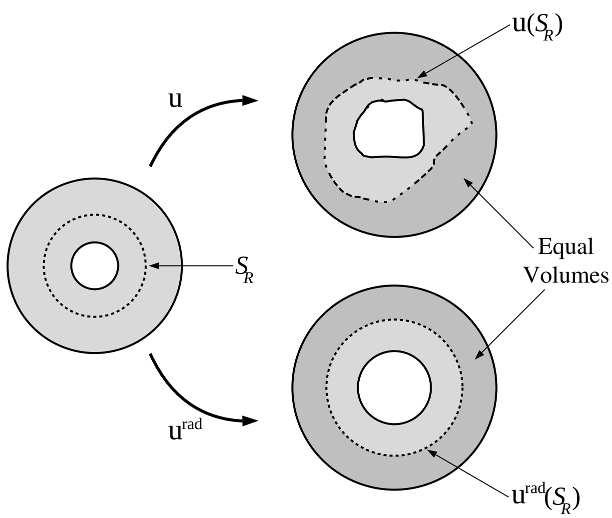

Figure 1: The radial symmetrisation $\mathbf{u}^{\mathrm{rad}}$ of $\mathbf{u}$.

As is illustrated in Figure 1, the radial symmetrisation thus replaces the deformed surface $\mathbf{u}(S_R)$, where S_R denotes the sphere of radius $R \in [a, b]$ centred at the origin, by a

[1] We obtain a corresponding result in the case $\mu < \lambda$ in Section 6 for energy functions whose involution (see Ball [2, pp. 210–211]) is of the form (1.2) or (1.3).

sphere of radius $r(R)$ which encloses the same volume.[2] It is then an important consequence of the classical isoperimetric inequality that the radial symmetrisation therefore satisfies

$$\text{Area}\big(\mathbf{u}(S_R)\big) \geq \text{Area}\big(\mathbf{u}^{\text{rad}}(S_R)\big). \tag{1.5}$$

Differentiating the expression $(1.4)_2$ with respect to R and dividing by the area of the sphere S_R we obtain

$$\fint_{S_R} (\det \nabla\mathbf{u})\, d\mathcal{H}^2_{\mathbf{x}} = \fint_{S_R} (\det \nabla\mathbf{u}^{\text{rad}})\, d\mathcal{H}^2_{\mathbf{x}} = \det \nabla\mathbf{u}^{\text{rad}} = r'(R)\left(\frac{r(R)}{R}\right)^2, \tag{1.6}$$

so that the radial symmetrisation preserves the average value of $\det \nabla\mathbf{u}$ on the spheres S_R. Using (1.4)–(1.6) we prove in Propositions 3.6 and 3.13 that if the radial symmetrisation $\mathbf{u}^{\text{rad}}$ satisfies

$$\frac{r(R)}{R} \geq r'(R) \quad \text{for some} \quad R \in (a,b), \tag{1.7}$$

then

$$\fint_{S_R} |\nabla\mathbf{u}|^3\, d\mathcal{H}^2_{\mathbf{x}} \geq \fint_{S_R} |\nabla\mathbf{u}^{\text{rad}}|^3\, d\mathcal{H}^2_{\mathbf{x}} = |\nabla\mathbf{u}^{\text{rad}}|^3 \tag{1.8}$$

and

$$\fint_{S_R} |\operatorname{adj} \nabla\mathbf{u}|^{\frac{3}{2}}\, d\mathcal{H}^2_{\mathbf{x}} \geq \fint_{S_R} |\operatorname{adj} \nabla\mathbf{u}^{\text{rad}}|^{\frac{3}{2}}\, d\mathcal{H}^2_{\mathbf{x}} = |\operatorname{adj} \nabla\mathbf{u}^{\text{rad}}|^{\frac{3}{2}} \tag{1.9}$$

for that same value of R. The proof of these inequalities bounds the left-hand sides of (1.8) and (1.9) from below by a convex function of the area of $\mathbf{u}(S_R)$ and the average (1.6). This function coincides with the right-hand sides of (1.8) and (1.9) when evaluated on the radial symmetrisation $\mathbf{u}^{\text{rad}}$ of $\mathbf{u}$. Note also that if the stored-energy function is strongly elliptic, then any non-homogeneous, radially symmetric solution of the equilibrium equations of the form $(1.4)_1$ must satisfy either $r'(R) < r(R)/R$ for all $R \in (a,b)$ or $r'(R) > r(R)/R$ for all $R \in (a,b)$ (see, e.g., [4, 17]).

The condition (1.7) can be interpreted as follows: by the construction of the radial symmetrisation (1.4), the ratio of the volume enclosed inside the deformed sphere $\mathbf{u}(S_R)$ to the volume contained within S_R is given by

$$\left(\frac{r(R)}{R}\right)^3.$$

[2] To see this, first note that the definition (1.4) guarantees that the volumes indicated in Figure 1 are equal and hence the total volumes enclosed within the deformed surfaces are also equal. (This uses the fact that the two deformations are equal on the outer boundary of the annulus.)

The condition (1.7) then guarantees that this ratio of the deformed volume to the reference volume is decreasing. Thus it follows that (1.4), (1.8), and (1.9) will all hold for any deformation $\mathbf{u}$ whose radial symmetrisation $\mathbf{u}^{\mathrm{rad}}$ satisfies (1.7) for all $R \in (a, b)$. These results together with Jensen's inequality and the structural assumptions (1.3) on the stored-energy function imply

$$\int_{S_R} W(\nabla \mathbf{u})\, d\mathcal{H}^2_{\mathbf{x}} \geq \int_{S_R} W(\nabla \mathbf{u}^{\mathrm{rad}})\, d\mathcal{H}^2_{\mathbf{x}} \quad \text{for all } R \in (a, b)$$

and hence, integrating with respect to R yields (see Proposition 3.14 and Corollary 3.15)

$$E(\mathbf{u}) \geq E(\mathbf{u}^{\mathrm{rad}}).$$

Therefore, the energy of $\mathbf{u}$ is no less than that of the minimiser amongst purely radial deformations (which is unique by [17, Theorems 2.4 and 2.5]).

If the deformation $\mathbf{u}$ is such that its radial symmetrisation does not satisfy (1.7) for all $R \in (a, b)$, then we show that, for the pure displacement boundary-value problem, there is a *modified radial symmetrisation* $\widetilde{\mathbf{u}}^{\mathrm{rad}}$ with no more total energy than $\mathbf{u}$ which satisfies (1.7) for all $R \in (a, b)$. Roughly speaking, this is achieved by replacing (1.4) by a homogeneous deformation on any sub-annulus on which condition (1.7) fails (the key idea in this construction is contained in Section 4 and is subsequently applied in Section 5).

For the mixed displacement/zero traction boundary-value problems we are able to use the ideas outlined above to prove that radially symmetric equilibria are strong relative minimisers of the energy (see Corollary 5.17).

For clarity of presentation, we first restrict attention to the case of C^1 deformations in Sections 1–6 and in Sections 7–8 we extend the results to Sobolev deformations.

2 Deformations and their Radial Symmetrisation.

2.1 Radial Deformations.

Throughout this paper we fix $b > a > 0$ and let $A \subset \mathbb{R}^n$ be the annulus given by

$$A := \{\mathbf{x} \in \mathbb{R}^n : a < |\mathbf{x}| < b\}. \tag{2.1}$$

We call $\mathbf{w}^{\mathrm{r}} : A \to \mathbb{R}^n$ a (smooth) *radial deformation* of A if there is a function $\rho \in C^1([a, b])$ that satisfies $\rho(a) > 0$, $\rho' > 0$ on $[a, b]$, and, for every $\mathbf{x} \in A$,

$$\mathbf{w}^{\mathrm{r}}(\mathbf{x}) = \frac{\rho(R)}{R}\mathbf{x}, \qquad R := |\mathbf{x}|.$$

In this case it follows that $\mathbf{w}^{\mathrm{r}} \in C^1(\overline{A};\mathbb{R}^n)$ is one-to-one with derivative (see, e.g., [4])

$$\nabla\mathbf{w}^{\mathrm{r}}(\mathbf{x}) = \rho'(R)\left(\frac{\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|^2}\right) + \frac{\rho(R)}{R}\left(\mathbf{I} - \frac{\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|^2}\right) \tag{2.2}$$

for every $\mathbf{x} \in A$. Consequently,

$$\operatorname{adj}\big(\nabla\mathbf{w}^{\mathrm{r}}(\mathbf{x})\big) = \left(\frac{\rho(R)}{R}\right)^{n-1}\left(\frac{\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|^2}\right) + \rho'(R)\left(\frac{\rho(R)}{R}\right)^{n-2}\left(\mathbf{I} - \frac{\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|^2}\right), \tag{2.3}$$

$$\det\big(\nabla\mathbf{w}^{\mathrm{r}}(\mathbf{x})\big) = \rho'(R)\left(\frac{\rho(R)}{R}\right)^{(n-1)}, \tag{2.4}$$

$$|\nabla\mathbf{w}^{\mathrm{r}}|^2 = \operatorname{tr}\left[(\nabla\mathbf{w}^{\mathrm{r}})^{\mathrm{T}}\nabla\mathbf{w}^{\mathrm{r}}\right] = \left[\rho'(R)\right]^2 + (n-1)\left(\frac{\rho(R)}{R}\right)^2, \tag{2.5}$$

$$\left|\operatorname{adj}\big(\nabla\mathbf{w}^{\mathrm{r}}\big)\right|^2 = \left(\frac{\rho(R)}{R}\right)^{2(n-1)} + (n-1)\left[\rho'(R)\right]^2\left(\frac{\rho(R)}{R}\right)^{2(n-2)}. \tag{2.6}$$

2.2 The Radial Symmetrisation.

Definition 2.1. Let $A \subset \mathbb{R}^n$ be the annulus given by (2.1) with inner boundary and outer boundary given, respectively, by

$$\partial A_I = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = a\}, \qquad \partial A_o = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = b\}.$$

For the mixed problem where the inner boundary is left free we let $\lambda > 0$ and define the set of (smooth) *admissible deformations* by

$$\mathcal{A}_\lambda^O = \left\{\mathbf{u} \in C^1(\overline{A};\mathbb{R}^n) : \det\nabla\mathbf{u} > 0 \text{ and } \mathbf{u} \text{ is one-to-one on } \overline{A},\ \mathbf{u}(\mathbf{x}) = \lambda\mathbf{x} \text{ for } \mathbf{x} \in \partial A_o\right\}.$$

For the mixed problem where the outer boundary is left free we let $\mu > 0$ and define the set of *admissible deformations* by

$$\mathcal{A}_\mu^I = \left\{\mathbf{u} \in C^1(\overline{A};\mathbb{R}^n) : \det\nabla\mathbf{u} > 0 \text{ and } \mathbf{u} \text{ is one-to-one on } \overline{A},\ \mathbf{u}(\mathbf{x}) = \mu\mathbf{x} \text{ for } \mathbf{x} \in \partial A_I\right\}.$$

For the displacement problem we let $\lambda > 0$, $\mu > 0$, $\mu a < \lambda b$, and define the set of *admissible deformations* by

$$\mathcal{A}_\mu^\lambda = \mathcal{A}_\lambda^O \cap \mathcal{A}_\mu^I.$$

For simplicity of exposition we present all of our results for $\mathbf{u} \in \mathcal{A}_\lambda^O$ or $\mathbf{u} \in \mathcal{A}_\mu^\lambda$ and note that the proof for $\mathbf{u} \in \mathcal{A}_\mu^I$ is similar. For each $\mathbf{u} \in \mathcal{A}_\lambda^O$ we define its *radial symmetrisation* by

$$\mathbf{u}^{\mathrm{rad}}(\mathbf{x}) = \frac{r(R)}{R}\mathbf{x}, \qquad R := |\mathbf{x}|, \quad \omega_n r(R)^n := \omega_n\lambda^n b^n - \int_{B_b\setminus B_R}(\det\nabla\mathbf{u})\,d\mathbf{x}, \tag{2.7}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ ($\omega_2 = \pi$ and $\omega_3 = 4\pi/3$).

We note $\mathbf{u}^{\mathrm{rad}}$ is differentiable with derivative given by (2.2) (with $\rho = r$), where r' satisfies

$$n\omega_n r^{n-1} r' = \int_{S_R} (\det \nabla \mathbf{u})\, d\mathcal{H}^{n-1}_{\mathbf{x}}, \tag{2.8}$$

and hence by (2.4)

$$\fint_{S_R} \det \nabla \mathbf{u}^{\mathrm{rad}}\, d\mathcal{H}^{n-1}_{\mathbf{x}} = r' \left(\frac{r}{R}\right)^{n-1} = \fint_{S_R} (\det \nabla \mathbf{u})\, d\mathcal{H}^{n-1}_{\mathbf{x}}, \tag{2.9}$$

where S_R denotes the sphere of radius R centred at the origin, and $\mathcal{H}^{n-1}$ denotes $(n-1)$-dimensional surface (Hausdorff) measure. Hence, in particular, the radial symmetrisation (2.7) preserves the average value[3] of $\det \nabla \mathbf{u}$ on the spheres S_R. It follows from (2.9) that

$$\int_{B_b \setminus B_R} \big(\det \nabla \mathbf{u}^{\mathrm{rad}}\big)\, d\mathbf{x} = \int_{B_b \setminus B_R} (\det \nabla \mathbf{u})\, d\mathbf{x}. \tag{2.10}$$

Lemma 2.2. *If $\mathbf{u} \in \mathcal{A}^O_\lambda$ then its radial symmetrisation, $\mathbf{u}^{\mathrm{rad}}$, is also contained in $\mathcal{A}^O_\lambda$.*

Proof. The smoothness of $\mathbf{u}^{\mathrm{rad}}$ is clear from (2.7) as is the fact that $\mathbf{u}^{\mathrm{rad}}$ satisfies the required boundary condition on the outer boundary S_b. Since $\det \nabla \mathbf{u} > 0$ it follows from (2.8) that r, given by (2.7), satisfies

$$r'(R)[r(R)]^{n-1} > 0 \text{ for every } R \in [a, b], \tag{2.11}$$

and hence $r' > 0$ on $[a, b]$ since $r \in C^1([a, b])$. Thus r is strictly monotone increasing, $\mathbf{u}^{\mathrm{rad}}$ is one-to-one, and $\det \nabla \mathbf{u}^{\mathrm{rad}} > 0$ on $\overline{A}$. Finally, the intermediate-value theorem together with (2.11) and $r(b) = \lambda b > 0$ yields $r(a) > 0$. □

The next proposition will be central to the arguments in this paper. The result, which is a direct consequence of the isoperimetric inequality, shows that the radial map (2.7) has the property that, amongst all maps in $\mathcal{A}^O_\lambda$ that enclose the same volume as $\mathbf{u}(S_R)$, it minimises the deformed area of each sphere S_R, $R \in [a, b]$. The proof follows as in Part I [18, Proposition 3.2], with the aid of (2.10) (see, also, the proof of Lemma 7.5 in this manuscript).

Proposition 2.3. *Let $\lambda > 0$, $\mathbf{u} \in \mathcal{A}^O_\lambda$, and suppose that $\mathbf{u}^{\mathrm{rad}}$ is given by (2.7). Then, for each $R \in [a, b]$,*

$$\mathrm{Area}\big(\mathbf{u}(S_R)\big) \geq \mathrm{Area}\big(\mathbf{u}^{\mathrm{rad}}(S_R)\big).$$

Moreover, the above inequality is strict at any R at which $\mathbf{u}(S_R)$ is not a spherical shell.

[3] We use $\fint_U \phi(\mathbf{x})\, d\mathcal{H}^{n-1}_{\mathbf{x}}$ to denote the average value of ϕ over U, i.e., the integral of ϕ over U divided by $\mathcal{H}^{n-1}(U)$.

3 Energy Reduction by Symmetrisation Alone.

We first assume $n = 3$ so that we are considering deformations $\mathbf{u} : A \subset \mathbb{R}^3 \to \mathbb{R}^3$. In Section 3.3 we show that if the radial symmetrisation $\mathbf{u}^{\mathrm{rad}}$ (defined by (2.7)) of a deformation $\mathbf{u} \in \mathcal{A}_\lambda^O$ satisfies

$$\frac{r(R)}{R} \geq r'(R) \quad \text{for some} \quad R \in [a, b],$$

then $\mathbf{u}^{\mathrm{rad}}$ satisfies

$$\fint_{S_R} W(\nabla \mathbf{u})\, d\mathcal{H}^2_{\mathbf{x}} \geq \fint_{S_R} W(\nabla \mathbf{u}^{\mathrm{rad}})\, d\mathcal{H}^2_{\mathbf{x}}$$

for any stored-energy function of the form (1.3). This will follow once we prove the result in the special cases $W(\mathbf{F}) = |\mathbf{F}|^3$ (in Section 3.1) and $W(\mathbf{F}) = |\operatorname{adj} \mathbf{F}|^{\frac{3}{2}}$ (in Section 3.2). We refer to Remark 3.7 for the corresponding results in the case $n = 2$ and for stored-energy functions of the form (1.2).

3.1 The case $W(\mathbf{F}) = |\mathbf{F}|^3$.

In this section we restrict our attention to the stored-energy function $W(\mathbf{F}) = |\mathbf{F}|^3$. The first result is a simple consequence of the invariance of the Dirichlet energy under a change in coordinates. Again, a proof of this result can be found in Part I.

Lemma 3.1. *Let $\lambda > 0$, $\mathbf{u} \in \mathcal{A}_\lambda^O$, and $R \in [a, b]$. At each point $\mathbf{x} \in S_R$ let $\mathbf{n}, \mathbf{t_1}, \mathbf{t_2}$ denote a right-handed orthonormal basis with $\mathbf{n} = \frac{\mathbf{x}}{|\mathbf{x}|}$. Then for $\mathbf{x} \in S_R$*

$$|\nabla \mathbf{u}|^2 = |(\nabla \mathbf{u})\, \mathbf{n}|^2 + |(\nabla \mathbf{u})\, \mathbf{t_1}|^2 + |(\nabla \mathbf{u})\, \mathbf{t_2}|^2 .$$

Lemma 3.2. *For each $\mathbf{x} \in S_R$, $R \in [a, b]$, we have*

$$|\nabla \mathbf{u}|^2 \geq 2 \left|(\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}} \mathbf{n}\right| + \frac{[J_{\mathbf{u}}]^2}{|(\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}} \mathbf{n}|^2}, \tag{3.1}$$

where $J_{\mathbf{u}} := \det \nabla \mathbf{u}$ denotes the Jacobian of $\mathbf{u}$.

Proof. By the previous lemma and the Cauchy-Schwarz inequality

$$\begin{aligned} |\nabla \mathbf{u}|^2 &\geq |(\nabla \mathbf{u})\, \mathbf{n}|^2 + 2\, |(\nabla \mathbf{u})\, \mathbf{t_1}|\, |(\nabla \mathbf{u})\, \mathbf{t_2}| \\ &\geq |(\nabla \mathbf{u})\, \mathbf{n}|^2 + 2\, |(\nabla \mathbf{u})\, \mathbf{t_1} \times (\nabla \mathbf{u})\, \mathbf{t_2}| . \end{aligned} \tag{3.2}$$

However,

$$(\nabla \mathbf{u})\, \mathbf{t_1} \times (\nabla \mathbf{u})\, \mathbf{t_2} = (\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}} \mathbf{n}, \tag{3.3}$$

while the Cauchy-Schwarz inequality also implies

$$\det \nabla \mathbf{u} = (\nabla \mathbf{u})\mathbf{n} \cdot (\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}}\mathbf{n} \leq |(\nabla \mathbf{u})\mathbf{n}||(\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}}\mathbf{n}|, \tag{3.4}$$

which together with (3.2) and (3.3) proves the desired result. □

Lemma 3.3. *For $s > 0$ and $t > 0$ define*

$$G(s,t) = \left[2\sqrt[3]{t^2} + \frac{s^2}{\sqrt[3]{t^4}} \right]^{\frac{3}{2}}. \tag{3.5}$$

Then the mapping $(s,t) \mapsto G(s,t)$ is convex and, for each $s_0 > 0$, the mapping $t \mapsto G(s_0, t)$ is monotone increasing for $t \geq s_0$.

Proof. We note that

$$G_t = \frac{2(t^2 - s^2)}{t^3} \sqrt{2t^2 + s^2}, \qquad G(s,t) = t\phi\left(\frac{s}{t}\right), \quad \text{where} \quad \phi(\tau) = \left(2 + \tau^2\right)^{\frac{3}{2}}.$$

The monotonicity is now clear. A simple computation shows that ϕ'' is positive, which together with Lemma A.1 shows that G is convex. □

Lemma 3.4. *Let $\lambda > 0$ and $\mathbf{u} \in \mathcal{A}_\lambda^O$. Then for each $R \in [a,b]$ we have*

$$\fint_{S_R} |\nabla \mathbf{u}|^3 \, d\mathcal{H}^2_{\mathbf{x}} \geq \left(2 \left[\fint_{S_R} (A_{\mathbf{u}})^{\frac{3}{2}} \, d\mathcal{H}^2_{\mathbf{x}} \right]^{\frac{2}{3}} + \frac{\left[\fint_{S_R} J_{\mathbf{u}} \, d\mathcal{H}^2_{\mathbf{x}} \right]^2}{\left[\fint_{S_R} (A_{\mathbf{u}})^{\frac{3}{2}} \, d\mathcal{H}^2_{\mathbf{x}} \right]^{\frac{4}{3}}} \right)^{\frac{3}{2}}, \tag{3.6}$$

where $A_{\mathbf{u}} := |(\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}}\mathbf{n}|$.

Proof. Define $A_{\mathbf{u}}$ as above and $\widehat{A}_{\mathbf{u}} := [A_{\mathbf{u}}]^{3/2}$. If we then we take equation (3.1) to the three-halves power we find that

$$|\nabla \mathbf{u}|^3 \geq \left[2[\widehat{A}_{\mathbf{u}}]^{\frac{2}{3}} + \frac{[J_{\mathbf{u}}]^2}{[\widehat{A}_{\mathbf{u}}]^{\frac{4}{3}}} \right]^{\frac{3}{2}}. \tag{3.7}$$

Next, if we integrate (3.7) over S_R and apply Jensen's inequality to the convex function $(s,t) \mapsto G(s,t)$, given in Lemma 3.3, with $t = \widehat{A}_{\mathbf{u}} = |(\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}}\mathbf{n}|^{3/2}$ and $s = J_{\mathbf{u}}$ we arrive at (3.6). □

Lemma 3.5. *Let $\lambda > 0$, $q \in [1, \infty)$, and $\mathbf{u} \in \mathcal{A}_\lambda^O$. Then, for each $R \in [a, b]$,*

$$\fint_{S_R} \left|(\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}} \mathbf{n}\right|^q \geq \left(\fint_{S_R} \left|(\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}} \mathbf{n}\right| \right)^q \tag{3.8}$$

$$\geq \left(\fint_{S_R} \left|\left(\operatorname{adj} \nabla \mathbf{u}^{\mathrm{rad}}\right)^{\mathrm{T}} \mathbf{n}\right| \right)^q \tag{3.9}$$

$$= \left[\frac{r(R)}{R}\right]^{2q} = \fint_{S_R} \left|\left(\operatorname{adj} \nabla \mathbf{u}^{\mathrm{rad}}\right)^{\mathrm{T}} \mathbf{n}\right|^q. \tag{3.10}$$

Proof. The first inequality is clear for $q = 1$ and follows from Hölder's inequality for $q > 1$. The second inequality is an immediate consequence of Proposition 2.3 since, for any $\mathbf{v} \in \mathcal{A}_\lambda^O$ and $R \in [a, b]$,

$$\int_{S_R} \left|(\operatorname{adj} \nabla \mathbf{v})^{\mathrm{T}} \mathbf{n}\right| = \operatorname{Area}(\mathbf{v}(S_R)).$$

The final equalities follow from (2.3) with $n = 3$. □

Proposition 3.6. *Let $\lambda > 0$ and $\mathbf{u} \in \mathcal{A}_\lambda^O$. Suppose that $\mathbf{u}^{\mathrm{rad}}$ satisfies*

$$0 \geq \frac{d}{dR}\left[\frac{r(R)}{R}\right] = \frac{1}{R}\left[r'(R) - \frac{r(R)}{R}\right] \tag{3.11}$$

for some $R \in [a, b]$. Then

$$\fint_{S_R} |\nabla \mathbf{u}|^3 \, d\mathcal{H}^2_{\mathbf{x}} \geq \fint_{S_R} |\nabla \mathbf{u}^{\mathrm{rad}}|^3 \, d\mathcal{H}^2_{\mathbf{x}} = |\nabla \mathbf{u}^{\mathrm{rad}}|^3. \tag{3.12}$$

Proof. Let $R \in [a, b]$ be such that (3.11) is satisfied. Then in view of (2.9) and (3.11)

$$s_0 := \fint_{S_R} J_{\mathbf{u}} \, d\mathcal{H}^2_{\mathbf{x}} = r'(R)\left[\frac{r(R)}{R}\right]^2 \leq \left[\frac{r(R)}{R}\right]^3. \tag{3.13}$$

Next, by Lemma 3.3 the function $t \mapsto G(s_0, t)$ is monotone on $[\frac{r(R)^3}{R^3}, \infty)$ for the above choice of $s_0 = s_0(R)$. Thus, if we make use of (2.8), (3.13), Lemma 3.4, and Lemma 3.5 (with $q = \frac{3}{2}$), we find that

$$\fint_{S_R} |\nabla \mathbf{u}|^3 \geq G\left(r'(R)\left[\frac{r(R)}{R}\right]^2, \fint_{S_R} \left|(\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}} \mathbf{n}\right|^{\frac{3}{2}} \right)$$

$$\geq G\left(r'(R)\left[\frac{r(R)}{R}\right]^2, \left[\frac{r(R)}{R}\right]^3 \right). \tag{3.14}$$

However, by (2.5) and (3.5),

$$G\left(r'\left[\frac{r}{R}\right]^2, \left[\frac{r}{R}\right]^3\right) = \left[2\left[\frac{r}{R}\right]^2 + \frac{[r']^2\left[\frac{r}{R}\right]^4}{\left[\frac{r}{R}\right]^4}\right]^{\frac{3}{2}} = |\nabla \mathbf{u}^{\mathrm{rad}}|^3,$$

which together with (3.14) yields (3.12). □

Remark 3.7. In the two-dimensional case, $n = 2$, the result corresponding to Proposition 3.6 is that, for any $\mathbf{u} \in \mathcal{A}_\lambda^O$,

$$\fint_{S_R} |\nabla \mathbf{u}|^2 \, d\mathcal{H}^1_{\mathbf{x}} \geq \fint_{S_R} |\nabla \mathbf{u}^{\mathrm{rad}}|^2 \, d\mathcal{H}^1_{\mathbf{x}} = |\nabla \mathbf{u}^{\mathrm{rad}}|^2,$$

where $\mathbf{u}^{\mathrm{rad}}$ is given by (2.7) with $n = 2$. This follows by analogous arguments to the case $n = 3$ with the following modifications:

1. For $n = 2$ the estimate corresponding to Lemma 3.2 is that, for each $\mathbf{x} \in S_R$ and $R \in [a, b]$,
$$|\nabla \mathbf{u}|^2 \geq \left|(\mathrm{adj}\, \nabla \mathbf{u})^{\mathrm{T}} \mathbf{n}\right|^2 + \frac{[J_{\mathbf{u}}]^2}{|(\mathrm{adj}\, \nabla \mathbf{u})^{\mathrm{T}} \mathbf{n}|^2}.$$

2. The function G in Lemma 3.3 is replaced by the function
$$\widetilde{G}(s, t) = t + \frac{s^2}{t},$$
which is convex with $t \mapsto \widetilde{G}(s_0, t)$ monotone increasing for $t \geq s_0$.

3. In Lemma 3.5 we take $q = 2$ and also replace the exponent $2q$ by 2.

Remark 3.8. For $A \subset \mathbb{R}^n$ and $\mathbf{u} : A \to \mathbb{R}^n$ a slight modification of the argument in this subsection shows that, if $r'(R) \leq r(R)/R$ for some $R \in [a, b]$, then

$$\fint_{S_R} |\nabla \mathbf{u}|^p \, d\mathcal{H}^{n-1}_{\mathbf{x}} \geq \fint_{S_R} |\nabla \mathbf{u}^{\mathrm{rad}}|^p \, d\mathcal{H}^{n-1}_{\mathbf{x}} = |\nabla \mathbf{u}^{\mathrm{rad}}|^p.$$

for all $p \geq n$. However, for $p < n$ the function corresponding to G (see (3.5)) is not convex and so our proof does not apply.

3.2 The case $W(\mathbf{F}) = |\,\mathrm{adj}\, \mathbf{F}|^{\frac{3}{2}}$.

In this section we restrict our attention to the stored energy function $W(\mathbf{F}) = |\,\mathrm{adj}\, \mathbf{F}|^{\frac{3}{2}}$. The proof of our first result can be found in [18].

Lemma 3.9. *Let $\lambda > 0$, $\mathbf{u} \in \mathcal{A}_\lambda^O$, and $R \in [a,b]$. At each point $\mathbf{x} \in S_R$ let $\mathbf{n}, \mathbf{t_1}, \mathbf{t_2}$ denote a right-handed orthonormal basis with $\mathbf{n} = \frac{\mathbf{x}}{|\mathbf{x}|}$. Then, for each $\mathbf{x} \in S_R$,*

$$|\operatorname{adj} \nabla\mathbf{u}|^2 = \left|(\operatorname{adj} \nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right|^2 + \left|(\operatorname{adj} \nabla\mathbf{u})^{\mathrm{T}}\mathbf{t}_1\right|^2 + \left|(\operatorname{adj} \nabla\mathbf{u})^{\mathrm{T}}\mathbf{t}_2\right|^2.$$

Lemma 3.10. *For each $\mathbf{x} \in S_R$ and $R \in [a,b]$, we have*

$$|\operatorname{adj} \nabla\mathbf{u}|^2 \geq \left|(\operatorname{adj} \nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right|^2 + 2\frac{[J_\mathbf{u}]^2}{|(\operatorname{adj} \nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}|}. \tag{3.15}$$

Proof. In view of Lemma 3.9

$$|\operatorname{adj} \nabla\mathbf{u}|^2 \geq \left|(\operatorname{adj} \nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right|^2 + 2\left|(\operatorname{adj} \nabla\mathbf{u})^{\mathrm{T}}\mathbf{t}_1 \times (\operatorname{adj} \nabla\mathbf{u})^{\mathrm{T}}\mathbf{t}_2\right|. \tag{3.16}$$

However,

$$(\operatorname{adj} \nabla\mathbf{u})^{\mathrm{T}}\mathbf{t}_1 \times (\operatorname{adj} \nabla\mathbf{u})^{\mathrm{T}}\mathbf{t}_2 = \left[\operatorname{adj}(\operatorname{adj} \nabla\mathbf{u})^{\mathrm{T}}\right]^{\mathrm{T}}\mathbf{n}, \tag{3.17}$$

$$\left[\operatorname{adj}(\operatorname{adj} \nabla\mathbf{u})^{\mathrm{T}}\right]^{\mathrm{T}} = (\det \nabla\mathbf{u})\nabla\mathbf{u}, \tag{3.18}$$

and the Cauchy-Schwarz inequality implies (see (3.4))

$$J_\mathbf{u} \leq \left|(\nabla\mathbf{u})\mathbf{n}\right| \left|(\operatorname{adj} \nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right|. \tag{3.19}$$

Equations (3.16)–(3.19) then yield (3.15). □

Lemma 3.11. *For $s > 0$ and $t > 0$ let*

$$H(s,t) = \left(\sqrt[3]{t^4} + 2\frac{s^2}{\sqrt[3]{t^2}}\right)^{\frac{3}{4}}. \tag{3.20}$$

Then $(s,t) \mapsto H(s,t)$ is convex and, for each $s_0 > 0$, the mapping $t \mapsto H(s_0,t)$ is monotone increasing for $t \geq s_0$.

Proof. We note that

$$H_t = \frac{t^2 - s^2}{\sqrt{t^3}\left(\sqrt[4]{2t^2 + s^2}\right)}, \qquad H(s,t) = t\phi\left(\frac{s}{t}\right), \quad \text{where} \quad \phi(\tau) = \left(1 + 2\tau^2\right)^{\frac{3}{4}}.$$

The monotonicity is now clear. A simple computation shows that ϕ'' is positive, which together with Lemma A.1 shows that H is convex. □

Lemma 3.12. *Let $\lambda > 0$ and $\mathbf{u} \in \mathcal{A}^{O}_{\lambda}$. Then for each $R \in [a, b]$ we have*

$$\fint_{S_R} |\operatorname{adj}\nabla\mathbf{u}|^{\frac{3}{2}}\, d\mathcal{H}^2_{\mathbf{x}} \geq \left(\left[\fint_{S_R} (A_{\mathbf{u}})^{\frac{3}{2}}\, d\mathcal{H}^2_{\mathbf{x}} \right]^{\frac{4}{3}} + 2\frac{\left[\fint_{S_R} J_{\mathbf{u}}\, d\mathcal{H}^2_{\mathbf{x}}\right]^2}{\left[\fint_{S_R} (A_{\mathbf{u}})^{\frac{3}{2}}\, d\mathcal{H}^2_{\mathbf{x}}\right]^{\frac{2}{3}}} \right)^{\frac{3}{4}}, \tag{3.21}$$

where $A_{\mathbf{u}} := |(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}|$.

Proof. Define $A_{\mathbf{u}}$ as above and $\widehat{A}_{\mathbf{u}} := [A_{\mathbf{u}}]^{3/2}$. If we then we take equation (3.15) to the three-fourths power we find that

$$|\operatorname{adj}\nabla\mathbf{u}|^{\frac{3}{2}} \geq \left[2[\widehat{A}_{\mathbf{u}}]^{\frac{4}{3}} + \frac{[J_{\mathbf{u}}]^2}{[\widehat{A}_{\mathbf{u}}]^{\frac{2}{3}}} \right]^{\frac{3}{4}}. \tag{3.22}$$

Next, if we integrate (3.22) over S_R and apply Jensen's inequality to the convex function $(s, t) \mapsto H(s, t)$, given in Lemma 3.11, with $t = \widehat{A}_{\mathbf{u}} = |(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}|^{3/2}$ and $s = J_{\mathbf{u}}$ we arrive at (3.21). □

Proposition 3.13. *Let $\lambda > 0$ and $\mathbf{u} \in \mathcal{A}^{O}_{\lambda}$. Suppose that $\mathbf{u}$ satisfies*

$$0 \geq \frac{d}{dR}\left[\frac{r(R)}{R}\right] = \frac{1}{R}\left[r'(R) - \frac{r(R)}{R}\right] \tag{3.23}$$

for some $R \in [a, b]$. Then

$$\fint_{S_R} |\operatorname{adj}\nabla\mathbf{u}|^{\frac{3}{2}}\, d\mathcal{H}^2_{\mathbf{x}} \geq \fint_{S_R} |\operatorname{adj}\nabla\mathbf{u}^{\mathrm{rad}}|^{\frac{3}{2}}\, d\mathcal{H}^2_{\mathbf{x}} = |\operatorname{adj}\nabla\mathbf{u}^{\mathrm{rad}}|^{\frac{3}{2}}. \tag{3.24}$$

Proof. Let $R \in [a, b]$ be such that (3.23) is satisfied. Then in view of (2.9) and (3.23)

$$s_0 := \fint_{S_R} J_{\mathbf{u}}\, d\mathcal{H}^2_{\mathbf{x}} = r'(R)\left[\frac{r(R)}{R}\right]^2 \leq \left[\frac{r(R)}{R}\right]^3. \tag{3.25}$$

Next, by Lemma 3.11 the function $t \mapsto H(s_0, t)$ is monotone on $[\frac{r(R)^3}{R^3}, \infty)$ for the above choice of $s_0 = s_0(R)$. Thus, if we make use of (2.8), (3.25), and Lemma 3.5 (with $q = \frac{3}{2}$), and Lemma 3.12, we conclude that

$$\begin{aligned}\fint_{S_R} |\operatorname{adj}\nabla\mathbf{u}|^{\frac{3}{2}} &\geq H\left(r'(R)\left[\frac{r(R)}{R}\right]^2, \fint_{S_R} \left|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right|^{\frac{3}{2}} \right) \\ &\geq H\left(r'(R)\left[\frac{r(R)}{R}\right]^2, \left[\frac{r(R)}{R}\right]^3 \right).\end{aligned} \tag{3.26}$$

However, by (2.6) and (3.20),

$$H\left(r'\left[\frac{r}{R}\right]^2, \left[\frac{r}{R}\right]^3\right) = \left[\left[\frac{r}{R}\right]^4 + 2\frac{[r']^2\left[\frac{r}{R}\right]^4}{\left[\frac{r}{R}\right]^2}\right]^{\frac{3}{2}} = |\operatorname{adj}\nabla\mathbf{u}^{\mathrm{rad}}|^{\frac{3}{2}},$$

which together with (3.26) yields (3.24). □

3.3 The case of general $W(\mathbf{F})$

Combining Sections 3.1 and 3.2 we obtain the following general result on the energy-reducing properties of the radial-symmetrisation procedure (2.7).

Proposition 3.14. *Let $n = 3$ and let W be polyconvex of the form* (1.3) *(or $n = 2$ and W of the form* (1.2)*). Let $\mathbf{u} \in \mathcal{A}^O_\lambda$ and suppose that its radial symmetrisation $\mathbf{u}^{\mathrm{rad}}$ (given by* (2.7)*) satisfies*

$$\frac{r(R)}{R} \geq r'(R) \quad \textit{for some} \quad R \in [a, b]. \tag{3.27}$$

Then

$$\fint_{S_R} W(\nabla\mathbf{u})\, d\mathcal{H}^{n-1}_{\mathbf{x}} \geq \fint_{S_R} W(\nabla\mathbf{u}^{\mathrm{rad}})\, d\mathcal{H}^{n-1}_{\mathbf{x}} = W(\nabla\mathbf{u}^{\mathrm{rad}}). \tag{3.28}$$

Proof. Suppose that $n = 3$ and that W is given by (1.3). The proof of this result then follows immediately from (3.24), (3.12), and (2.9) by Jensen's inequality on noting that $|\nabla\mathbf{u}^{\mathrm{rad}}|^3$, $|\operatorname{adj}\nabla\mathbf{u}^{\mathrm{rad}}|^{\frac{3}{2}}$ and $\det\nabla\mathbf{u}^{\mathrm{rad}}$ are all functions of R only. (A similar argument holds in the case $n = 2$ for stored-energy functions of the form (1.2).) □

Corollary 3.15. *Suppose that $\mathbf{u} \in \mathcal{A}^O_\lambda$ is such that its radial symmetrisation $\mathbf{u}^{\mathrm{rad}}$ satisfies* (3.27) *for all $R \in (a, b)$. Then*

$$\int_A W(\nabla\mathbf{u})\, d\mathbf{x} \geq \int_A W(\nabla\mathbf{u}^{\mathrm{rad}})\, d\mathbf{x}.$$

Proof. The proof of this result follows from (3.28) on multiplying both sides by $4\pi R^2$ and integrating with respect to R on the interval $[a, b]$. □

4 Energy Reduction by Homogeneous Replacement.

In this section we again take $A \subset \mathbb{R}^n$, $n = 2$ or $n = 3$, and consider stored-energy functions of the form (1.2) or (1.3). Proposition 3.14 shows that the radial symmetrisation reduces the average energy on spheres whenever (3.27) holds. The results of this section will be used later in Section 5 to modify the radial symmetrisation on certain sub-annuli when the condition (3.27) does not hold for all $R \in (a, b)$. We show that if the radial symmetrisation, $\mathbf{u}^{\text{rad}}$, of $\mathbf{u} \in \mathcal{A}_\lambda^O$ satisfies[4]

$$\frac{1}{\alpha}\mathbf{u}^{\text{rad}}(\alpha\mathbf{n}) = \frac{1}{\beta}\mathbf{u}^{\text{rad}}(\beta\mathbf{n}) \quad \text{for all} \quad \mathbf{n} \in \mathbb{S}^{n-1}, \tag{4.1}$$

for some $a \leq \alpha < \beta \leq b$, then there is a homogeneous deformation that also satisfies (4.1) and whose energy on the sub-annulus $\alpha \leq |\mathbf{x}| \leq \beta$ is no greater than the energy of $\mathbf{u}$ on the given sub-annulus (see Corollary 4.2). Again, as in Section 3, by using Jensen's inequality, it suffices to restrict our attention to stored-energy functions of the special form $W(\mathbf{F}) = |\mathbf{F}|^n$, $W(\mathbf{F}) = |\operatorname{adj}\mathbf{F}|^{\frac{n}{n-1}}$, and $W(\mathbf{F}) = \det\mathbf{F}$.

Proposition 4.1. *Let $\lambda > 0$ and $\mathbf{u} \in \mathcal{A}_\lambda^O$. Suppose that $\mathbf{u}^{\text{rad}}$, the radial symmetrisation of $\mathbf{u}$ given by* (2.7)*, satisfies*

$$\frac{r(\alpha)}{\alpha} = \frac{r(\beta)}{\beta} =: \sigma, \tag{4.2}$$

where $a \leq \alpha < \beta \leq b$. Define $\mathbf{u}_\sigma(\mathbf{x}) := \sigma\mathbf{x}$. Then

$$⨍_{B_\beta \setminus B_\alpha} |\nabla\mathbf{u}|^n \, d\mathbf{x} \geq ⨍_{B_\beta \setminus B_\alpha} |\nabla\mathbf{u}_\sigma|^n \, d\mathbf{x} = |\nabla\mathbf{u}_\sigma|^n, \tag{4.3}$$

$$⨍_{B_\beta \setminus B_\alpha} |\operatorname{adj}\nabla\mathbf{u}|^{\frac{n}{n-1}} \, d\mathbf{x} \geq ⨍_{B_\beta \setminus B_\alpha} |\operatorname{adj}\nabla\mathbf{u}_\sigma|^{\frac{n}{n-1}} \, d\mathbf{x} = |\operatorname{adj}\nabla\mathbf{u}_\sigma|^{\frac{n}{n-1}}, \tag{4.4}$$

$$⨍_{B_\beta \setminus B_\alpha} \det\nabla\mathbf{u} \, d\mathbf{x} = ⨍_{B_\beta \setminus B_\alpha} \det\nabla\mathbf{u}_\sigma \, d\mathbf{x} = \det\nabla\mathbf{u}_\sigma. \tag{4.5}$$

Proof. Let $\lambda > 0$, $\mathbf{u} \in \mathcal{A}_\lambda^O$, and $\mathbf{u}^{\text{rad}}$ be given by (2.7). We first note that (2.7) and (4.2) imply

$$⨍_{B_\beta \setminus B_\alpha} (\det\nabla\mathbf{u}) \, d\mathbf{x} = \frac{r(\beta)^n - r(\alpha)^n}{\beta^n - \alpha^n} = \sigma^n. \tag{4.6}$$

However, $\nabla\mathbf{u}_\sigma = \sigma\mathbf{I}$ and hence $\det\nabla\mathbf{u}_\sigma = \sigma^n$, which together with (4.6) yields (4.5).

Next, by the arithmetic-geometric mean inequality, for all $\nu_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$,

$$\frac{1}{n}\sum_{i=1}^n \nu_i^2 \geq \left[\prod_{i=1}^n \nu_i^2\right]^{1/n} \text{ with equality if and only if } \nu_1 = \nu_2 = \ldots = \nu_n.$$

In particular, if one chooses ν_i^2 to be the eigenvalues of $\mathbf{F}\mathbf{F}^{\mathrm{T}}$ for any $\mathbf{F} \in \mathrm{M}^{n\times n}$ one concludes

$$|\mathbf{F}|^n \geq n^{(n/2)}|\det\mathbf{F}|. \tag{4.7}$$

[4]Here $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ is the set of unit vectors, i.e., the unit sphere.

If we substitute $\mathbf{F} = \nabla \mathbf{u}$ in (4.7) and integrate over the set $B_\beta \backslash B_\alpha$ we find, with the aid of (4.6), that

$$\frac{1}{n^{(n/2)}} \fint_{B_\beta \backslash B_\alpha} |\nabla \mathbf{u}|^n \, d\mathbf{x} \geq \fint_{B_\beta \backslash B_\alpha} \det \nabla \mathbf{u} \, d\mathbf{x} = \sigma^n. \tag{4.8}$$

We note that $|\nabla \mathbf{u}_\sigma|^n = n^{(n/2)} \sigma^n$, which together with (4.8) implies (4.3).

To prove (4.4) we first take $\mathbf{G} \in \mathrm{M}^{n \times n}$ and let $\mathbf{F} = \operatorname{adj} \mathbf{G}$ in (4.7) to conclude

$$|\operatorname{adj} \mathbf{G}|^n \geq n^{(n/2)} |\det \mathbf{G}|^{n-1}. \tag{4.9}$$

Now let $\mathbf{G} = \nabla \mathbf{u}$ in (4.9) and integrate over the set $B_\beta \backslash B_\alpha$ to conclude, with the aid of (4.6), that

$$n^{\frac{n}{2-2n}} \fint_{B_\beta \backslash B_\alpha} |\operatorname{adj} \nabla \mathbf{u}|^{\frac{n}{n-1}} \, d\mathbf{x} \geq \fint_{B_\beta \backslash B_\alpha} \det \nabla \mathbf{u} \, d\mathbf{x} = \sigma^n. \tag{4.10}$$

We note that $\operatorname{adj} \nabla \mathbf{u}_\sigma = \sigma^{(n-1)} \mathbf{I}$ and hence $|\operatorname{adj} \nabla \mathbf{u}_\sigma|^n = n^{(n/2)} \sigma^{n(n-1)}$, which together with (4.10) yields (4.4). □

Corollary 4.2. *Let $n = 3$ and let W be of the form* (1.3) *(or $n = 2$ and W of the form* (1.2)*). Suppose that $\mathbf{u} \in \mathcal{A}_\lambda^O$ satisfies the hypotheses of Proposition 4.1. Then*

$$\int_{B_\beta \backslash B_\alpha} W(\nabla \mathbf{u}) \, d\mathbf{x} \geq \int_{B_\beta \backslash B_\alpha} W(\nabla \mathbf{u}_\sigma) \, d\mathbf{x}.$$

Proof. We prove this result in the case $n = 3$ (the case $n = 2$ is similar). It follows from (4.3)–(4.5) and Jensen's inequality that

$$\fint_{B_\beta \backslash B_\alpha} \Phi(|\nabla \mathbf{u}|^3, |\operatorname{adj} \nabla \mathbf{u}|^{\frac{3}{2}}, \det \nabla \mathbf{u}) \, d\mathbf{x} \geq \Phi(|\nabla \mathbf{u}_\sigma|^3, |\operatorname{adj} \nabla \mathbf{u}_\sigma|^{\frac{3}{2}}, \det \nabla \mathbf{u}_\sigma);$$

the result now follows on multiplying both sides by the volume of $B_\beta \backslash B_\alpha$ and noting that $\nabla \mathbf{u}_\sigma = \sigma \mathbf{I}$ is constant on $B_\beta \backslash B_\alpha$. □

5 Global Energy Reduction.

In Proposition 3.14 we demonstrated that for each map $\mathbf{u} \in \mathcal{A}_\lambda^O$ the symmetrisation procedure (2.7) produces a radial deformation $\mathbf{u}^{\mathrm{rad}}$ whose average energy on each sphere S_R is no greater than that of $\mathbf{u}$ at each value of $R \in (a, b)$ for which the condition $r'(R) \leq \frac{r(R)}{R}$ holds. In this section we combine the arguments of Sections 3 and 4 in order to construct a modified radial symmetrisation $\widetilde{\mathbf{u}}^{\mathrm{rad}}$ that has less energy than $\mathbf{u}$ on the entire annulus even if the condition $r'(R) \leq \frac{r(R)}{R}$ is not satisfied for all $R \in (a, b)$. The key is the following result, which we prove in the appendix.

Lemma 5.1. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous. Define $g : [a,b] \to \mathbb{R}$ by*

$$g(x) := \inf_{t\in[a,x]} f(t). \tag{5.1}$$

Then g is absolutely continuous and monotone decreasing. Moreover, there exist a countable sequence of pairwise-disjoint open intervals, $(a_k, b_k) \subset (a, b)$, such that

I. $g \equiv g_k$ is constant on each interval (a_k, b_k), where $g_k \leq f(a)$;

II. $g = f$ on $K := (a, b) \setminus \cup_k (a_k, b_k)$; and

III. $g' = f' \chi_K$ a.e. on $[a, b]$.

Additionally, if f is Lipschitz continuous then so is g.

Remark 5.2. In the sequel we will make use of the function

$$\widetilde{g}(x) = \max\{g(x), f(b)\}. \tag{5.2}$$

It is clear from the proof of Lemma 5.1 that $\widetilde{g}$ has the same regularity properties as g and that $f(a) \geq \widetilde{g}(x) \geq f(b)$ for all $x \in [a, b]$.

5.1 Radial Deformations Revisited.

The mapping $\widetilde{g}$ given by (5.1) and (5.2) is not, in general, C^1 even when f is analytic. If the number of intervals (a_k, b_k) happens to be finite then $\widetilde{g}$ is continuous and piecewise C^1. Otherwise, $\widetilde{g}$ is only Lipschitz continuous. However, the classical theory for absolutely continuous functions on the real line shows that such functions are differentiable almost everywhere. In addition absolutely continuous functions satisfy the fundamental theorem of calculus. We will next extend our class of radial deformations to include such functions.

Definition 5.3. Let $A \subset \mathbb{R}^n$ and $1 \leq p \leq \infty$. We call a mapping $\mathbf{w}^{\mathrm{r}} \in W^{1,p}(A; \mathbb{R}^n)$ a *radial Sobolev deformation* of A if there is an absolutely continuous, strictly monotone increasing function $\rho : [a, b] \to \mathbb{R}$ that satisfies $\rho(a) > 0$ and

$$\mathbf{w}^{\mathrm{r}}(\mathbf{x}) = \frac{\rho(R)}{R}\mathbf{x}, \qquad R := |\mathbf{x}| \tag{5.3}$$

for almost every $\mathbf{x} \in A$. Here $W^{1,p}(A; \mathbb{R}^n)$ denotes the usual Sobolev space of mappings $\mathbf{u} \in \mathcal{L}^p(A; \mathbb{R}^n)$ whose distributional derivative is also contained in $\mathcal{L}^p$.

Remark 5.4. Since ρ is strictly monotone $\mathbf{w}^{\mathrm{r}}$ is one-to-one.

Remark 5.5. Suppose, for the moment, that the stored-energy function W is continuous, polyconvex and satisfies standard growth conditions. Then results of Ball [1] yield the existence of a minimiser of the elastic energy among those elements of $W^{1,p}(A;\mathbb{R}^n)$, for appropriate p, that are locally invertible and satisfy suitable displacement boundary conditions. If these boundary conditions are radial then the analysis in, e.g., [1, 4, 17] yields the existence of a minimiser among radial Sobolev deformations. Moreover, additional smoothness and convexity hypotheses on the energy then imply (see, e.g. [4]) the regularity of the radial minimiser, i.e., the radial minimiser is in fact a *smooth* radial deformation. Thus, although we have extended our notion of a deformation, we expect that there is a smooth radial deformation that is an absolute minimiser of the energy.

We now recall a result of [4] concerning the regularity of radial deformations.

Lemma 5.6. *Let $p \in [1,\infty)$. Suppose that $\mathbf{w}^{\mathrm{r}}$ satisfies* (5.3) *for a.e. $\mathbf{x} \in A$. Then $\mathbf{w}^{\mathrm{r}} \in W^{1,p}(A;\mathbb{R}^n)$ if and only if ρ is absolutely continuous on $[a,b]$ with $\rho' \in \mathcal{L}^p((a,b))$. Moreover, the distributional derivative of $\mathbf{w}^{\mathrm{r}}$ is given by* (2.2) *and* (2.3)–(2.4) *are satisfied for a.e. $\mathbf{x}$.*

Remark 5.7. If $\rho'(R) > 0$ for $\mathcal{L}^1$ a.e. $R \in (a,b)$ then $\mathbf{w}^{\mathrm{r}}$ is one-to-one and satisfies $\det \nabla \mathbf{w}^{\mathrm{r}}(\mathbf{x}) > 0$ for $\mathcal{L}^n$ a.e. $\mathbf{x} \in A$.

Definition 5.8. For the mixed problem where the outer boundary is left free we let $\lambda > 0$ and define the set of (absolutely continuous) *admissible radial deformations* by

$$\mathcal{R}_\lambda^O := \left\{ \mathbf{w}^{\mathrm{r}}(\mathbf{x}) = \frac{\rho(R)}{R}\mathbf{x} \in W^{1,1}(A;\mathbb{R}^n) : \begin{array}{c} \rho(a) > 0, \quad \rho' > 0 \text{ a.e.}, \\ \mathbf{w}^{\mathrm{r}}(\mathbf{x}) = \lambda \mathbf{x} \text{ for } \mathbf{x} \in \partial A_o \end{array} \right\}.$$

For the displacement problem we let $\lambda > 0$, $\mu > 0$, $\mu a < \lambda b$, and define the set of *admissible radial deformations* by

$$\mathcal{R}_\mu^\lambda := \left\{ \mathbf{w}^{\mathrm{r}} \in \mathcal{R}_\lambda^O : \mathbf{w}^{\mathrm{r}}(\mathbf{x}) = \mu \mathbf{x} \text{ for } \mathbf{x} \in \partial A_I \right\}.$$

5.2 The stored-energy functions $|\mathbf{F}|^3$, $|\operatorname{adj}\mathbf{F}|^{\frac{3}{2}}$, and $\det \mathbf{F}$.

For the remainder of the section we will let $n = 3$. In this subsection we consider the stored-energy functions $W(\mathbf{F}) = |\mathbf{F}|^3$, $W(\mathbf{F}) = |\operatorname{adj}\mathbf{F}|^{\frac{3}{2}}$, and $W(\mathbf{F}) = \det \mathbf{F}$. We obtain a condition on deformations $\mathbf{u} \in \mathcal{A}_\lambda^O$ that allows us to construct a radial deformation $\widetilde{\mathbf{u}}^{\mathrm{rad}} \in \mathcal{R}_\lambda^O$ of lower energy.

Proposition 5.9. *Let $\lambda > 0$, $\mathbf{u} \in \mathcal{A}_\lambda^O$ and suppose that $\mathbf{u}^{\mathrm{rad}}$ is the radial symmetrisation of $\mathbf{u}$ given by* (2.7). *Assume that $\mathbf{u}^{\mathrm{rad}}$ satisfies*

$$\mu := \frac{r(a)}{a} > \frac{r(b)}{b} = \lambda. \tag{5.4}$$

Then there exists pairwise-disjoint open intervals $(a_k, b_k) \subset [a, b]$ *and a radial deformation*

$$\widetilde{\mathbf{u}}^{\mathrm{rad}}(\mathbf{x}) = \frac{\rho(R)}{R}\mathbf{x} \in \mathcal{R}^{\lambda}_{\mu},$$

which we refer to as the modified radial symmetrisation, such that

(i) $\rho : [a, b] \to [\mu a, \lambda b]$ *is bi-Lipschitz;*

(ii) $\rho(R) = g_k R$ *for* $R \in [a_k, b_k]$ *where* $\widetilde{\mathbf{u}}^{\mathrm{rad}}(\mathbf{x}) = g_k \mathbf{x}$ *for* $|\mathbf{x}| = a_k$ *and also for* $|\mathbf{x}| = b_k$*;*

(iii) For almost every $R \in K := [a, b] \setminus \bigcup_k (a_k, b_k)$

$$\rho(R) = r(R), \qquad \rho'(R) = r'(R), \tag{5.5}$$

and hence

$$0 \geq \frac{d}{dR}\left[\frac{\rho(R)}{R}\right] = \frac{1}{R}\left[\rho'(R) - \frac{\rho(R)}{R}\right];$$

(iv) For each k

$$\int_{a_k}^{b_k} \int_{S_R} |\nabla \mathbf{u}|^3 \, d\mathcal{H}^2_{\mathbf{x}} \, dR \geq \int_{a_k}^{b_k} \int_{S_R} |\nabla \widetilde{\mathbf{u}}^{\mathrm{rad}}|^3 \, d\mathcal{H}^2_{\mathbf{x}} \, dR, \tag{5.6}$$

$$\int_{a_k}^{b_k} \int_{S_R} |\operatorname{adj} \nabla \mathbf{u}|^{\frac{3}{2}} \, d\mathcal{H}^2_{\mathbf{x}} \, dR \geq \int_{a_k}^{b_k} \int_{S_R} |\operatorname{adj} \nabla \widetilde{\mathbf{u}}^{\mathrm{rad}}|^{\frac{3}{2}} \, d\mathcal{H}^2_{\mathbf{x}} \, dR, \tag{5.7}$$

$$\int_{a_k}^{b_k} \int_{S_R} \big(\det \nabla \mathbf{u}\big) \, d\mathcal{H}^2_{\mathbf{x}} \, dR = \int_{a_k}^{b_k} \int_{S_R} \big(\det \nabla \widetilde{\mathbf{u}}^{\mathrm{rad}}\big) \, d\mathcal{H}^2_{\mathbf{x}} \, dR. \tag{5.8}$$

Remark 5.10. By *bi-Lipschitz* we mean that $\rho : [a, b] \to [\mu a, \lambda b]$ is Lipschitz continuous with inverse $\rho^{-1} : [\mu a, \lambda b] \to [a, b]$ that is also Lipschitz continuous. Given $\rho' > 0$ a.e., the hypothesis that the inverse is Lipschitz continuous is equivalent to the assumption that there is an $\epsilon > 0$ such that $\rho' \geq \epsilon$ a.e. on $[a, b]$, as can be seen by applying the fundamental theorem of calculus. The Lipschitz constant for ρ^{-1} is then $1/\epsilon$.

Proof of Proposition 5.9. Let $\lambda > 0$ and $\mathbf{u} \in \mathcal{A}^{O}_{\lambda}$. Suppose that $\mathbf{u}^{\mathrm{rad}}$ and r are given by (2.7). Then, since $\mathbf{u}$ is C^1 with $\det \nabla \mathbf{u} > 0$ on $\overline{A}$, it follows from (2.8) that

$$r' \geq \epsilon_0 > 0 \text{ on } [a, b] \tag{5.9}$$

for some $\epsilon_0 > 0$. We next define $f \in C^1([a, b])$ by

$$f(R) = \frac{r(R)}{R}.$$

Then by Lemma 5.1 and the remark that follows it the function $\widetilde{g} : [a, b] \to \mathbb{R}$ given by (5.1) and (5.2) is Lipschitz continuous and satisfies properties I–III of Lemma 5.1. Next, define $\rho : [a, b] \to \mathbb{R}$ and $\widetilde{\mathbf{u}}^{\mathrm{rad}} : \overline{A} \to \mathbb{R}^3$ by

$$\rho(R) = R\,\widetilde{g}(R), \qquad \widetilde{\mathbf{u}}^{\mathrm{rad}}(\mathbf{x}) = \frac{\rho(R)}{R}\mathbf{x},$$

respectively. Then ρ is Lipschitz continuous and satisfies property I of Lemma 5.1 (and $R \mapsto \rho(R)/R$ satisfies II and III). Moreover, by Lemma 5.6, $\widetilde{\mathbf{u}}^{\mathrm{rad}} \in W^{1,1}(A; \mathbb{R}^3)$ with (distributional) derivative given by (2.2). In addition, $\rho(a) = r(a) = \mu a$ and $\rho(b) = r(b) = \lambda b$. Thus, in view of Remark 5.10, once we establish that $\rho' \geq \epsilon > 0$ a.e. it will follow that $\widetilde{\mathbf{u}}^{\mathrm{rad}}$ is bi-Lipschitz.

We now consider $R \in K$. For almost every $R \in K$ we have $f(R) = g(R)$, $f'(R) = g'(R)$, and consequently $r(R) = \rho(R)$ and $r'(R) = \rho'(R)$. Then since $\widetilde{g}$ is monotone decreasing

$$0 \geq \widetilde{g}'(R) = \frac{d}{dR}\left[\frac{\rho(R)}{R}\right] = \frac{1}{R}\left(\rho'(R) - \frac{\rho(R)}{R}\right),$$

which establishes (iii). In addition, since $r' = \rho'$ a.e. on K it follows from (5.9) that $\rho' \geq \epsilon_0 > 0$ a.e. on K.

Next let $R \in (a_k, b_k)$. Since $\widetilde{g}(R) \equiv g_k$, a constant, on $[a_k, b_k]$ and $a_k, b_k \in K$ we find that

$$g_k = \frac{r(a_k)}{a_k} = \frac{\rho(R)}{R} = \frac{r(b_k)}{b_k} \quad \text{for all} \quad R \in [a_k, b_k], \tag{5.10}$$

which establishes (ii). In addition, (5.10) implies that $\rho'(R) = g_k > 0$ for all $R \in [a_k, b_k]$. Thus since the g_k are bounded away from zero we have completed the proof that $\rho' \geq \epsilon$ a.e., which in view of Remark 5.10, establishes (i).

Finally, (5.10), Proposition 4.1, and Fubini's theorem yield (5.6)–(5.8), which establishes part (iv) of the proposition. □

Remark 5.11. Suppose that $\mathbf{u} \in \mathcal{A}_\lambda^O$ and that $\mathbf{u}^{\mathrm{rad}}$ is the radial symmetrisation of $\mathbf{u}$ given by (2.7). The idea for the modification, $\widetilde{\mathbf{u}}^{\mathrm{rad}}$, of the radial symmetrisation in Proposition 5.9 is to replace $\mathbf{u}^{\mathrm{rad}}$ by a homogeneous deformation on any sub-annulus in which $\frac{r(R)}{R}$ is not monotone decreasing: see figures 2 and 3 for an illustration of this construction.

5.3 The general case: $W(\mathbf{F}) = \Phi(|\mathbf{F}|^3, |\operatorname{adj}\mathbf{F}|^{\frac{3}{2}}, \det\mathbf{F})$.

We start with the pure displacement problem where we obtain the strongest result.

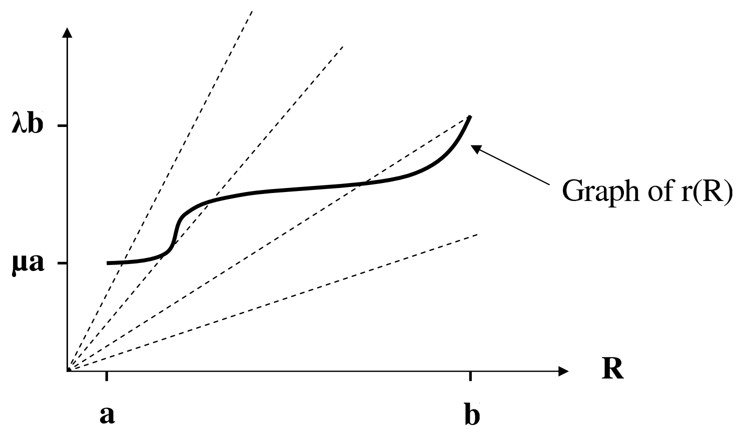

Figure 2: Graph of radial symmetrisation $\mathbf{u}^{\text{rad}}$.

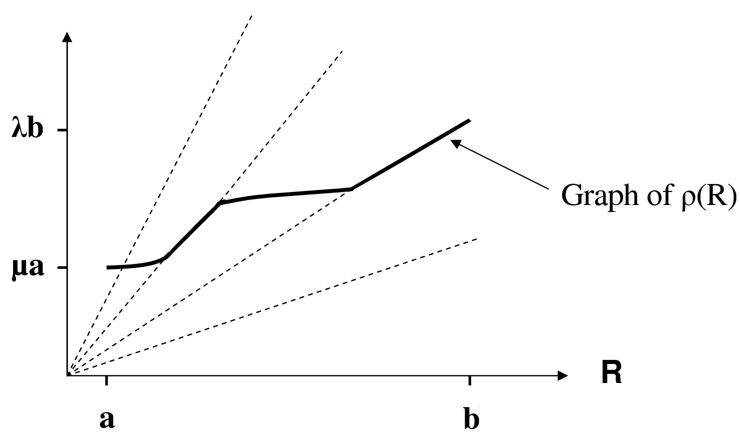

Figure 3: Graph of modified radial symmetrisation $\widetilde{\mathbf{u}}^{\text{rad}}$.

Theorem 5.12. *Let $0 < \mu a < \lambda b$ with $\mu \geq \lambda$. Suppose that $\mathbf{u} \in \mathcal{A}_\mu^\lambda$ and*

$$W(\mathbf{F}) = \Phi(|\mathbf{F}|^3, |\operatorname{adj}\mathbf{F}|^{\frac{3}{2}}, \det\mathbf{F}), \tag{5.11}$$

where $\Phi : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is convex and $s \mapsto \Phi(s, t, j)$ and $s \mapsto \Phi(t, s, j)$ are increasing functions for all $t > 0$ and $j > 0$. Then there exists a bi-Lipschitz radial deformation $\widetilde{\mathbf{u}}^{\mathrm{rad}} \in \mathcal{R}_\mu^\lambda$ that satisfies

$$E(\mathbf{u}) = \int_A W(\nabla\mathbf{u})\,d\mathbf{x} \geq \int_A W(\nabla\widetilde{\mathbf{u}}^{\mathrm{rad}})\,d\mathbf{x} = E(\widetilde{\mathbf{u}}^{\mathrm{rad}}).$$

Proof. If $\mu = \lambda$ then the quasiconvexity of the integrand implies that $\widetilde{\mathbf{u}}^{\mathrm{rad}}(\mathbf{x}) \equiv \lambda\mathbf{x}$ is a global minimiser of the energy. Otherwise, $\mu > \lambda$ and so we can apply Proposition 5.9 to conclude that there exists pairwise-disjoint open intervals $(a_k, b_k) \subset [a, b]$ and a bi-Lipschitz radial deformation

$$\widetilde{\mathbf{u}}^{\mathrm{rad}}(\mathbf{x}) = \frac{\rho(R)}{R}\mathbf{x} \in \mathcal{R}_\mu^\lambda$$

that satisfy (5.6)–(5.8), $\rho(R) = g_k R$ for $R \in [a_k, b_k]$, where $r(a_k) = g_k a_k$ and $r(b_k) = g_k b_k$, and for a.e. $R \in K := [a, b]\setminus\bigcup_k(a_k, b_k)$, $\rho(R) = r(R)$, $\rho'(R) = r'(R)$, and

$$0 \geq \frac{d}{dR}\left[\frac{\rho(R)}{R}\right] = \frac{1}{R}\left[\rho'(R) - \frac{\rho(R)}{R}\right]. \tag{5.12}$$

We next compute the energy of $\widetilde{\mathbf{u}}^{\mathrm{rad}}$ and compare it to the energy of $\mathbf{u}$. First consider $R \in K$. Since (5.12) is satisfied for almost every $R \in K$, Proposition 3.6 and Proposition 3.13 together with (2.4) and (5.5) imply that, for such R,

$$\fint_{S_R} |\nabla\mathbf{u}|^3\,d\mathcal{H}^2_{\mathbf{x}} \geq \fint_{S_R} |\nabla\mathbf{u}^{\mathrm{rad}}|^3\,d\mathcal{H}^2_{\mathbf{x}} = \fint_{S_R} |\nabla\widetilde{\mathbf{u}}^{\mathrm{rad}}|^3\,d\mathcal{H}^2_{\mathbf{x}}, \tag{5.13}$$

$$\fint_{S_R} |\operatorname{adj}\nabla\mathbf{u}|^{\frac{3}{2}}\,d\mathcal{H}^2_{\mathbf{x}} \geq \fint_{S_R} |\operatorname{adj}\nabla\mathbf{u}^{\mathrm{rad}}|^{\frac{3}{2}}\,d\mathcal{H}^2_{\mathbf{x}} = \fint_{S_R} |\operatorname{adj}\nabla\widetilde{\mathbf{u}}^{\mathrm{rad}}|^{\frac{3}{2}}\,d\mathcal{H}^2_{\mathbf{x}}, \tag{5.14}$$

$$\fint_{S_R} \det\nabla\mathbf{u}\,d\mathcal{H}^2_{\mathbf{x}} = \fint_{S_R} \det\nabla\mathbf{u}^{\mathrm{rad}}\,d\mathcal{H}^2_{\mathbf{x}} = \fint_{S_R} \det\nabla\widetilde{\mathbf{u}}^{\mathrm{rad}}\,d\mathcal{H}^2_{\mathbf{x}}. \tag{5.15}$$

Consequently, Jensen's inequality and the monotonicity of Φ in its first two arguments yield

$$\begin{aligned}
&\fint_{S_R} \Phi(|\nabla\mathbf{u}|^3, |\operatorname{adj}\nabla\mathbf{u}|^{\frac{3}{2}}, \det\nabla\mathbf{u})\,d\mathcal{H}^2_{\mathbf{x}} \\
&\qquad\geq \Phi\left(\fint_{S_R} |\nabla\mathbf{u}|^3\,d\mathcal{H}^2_{\mathbf{x}}, \fint_{S_R} |\operatorname{adj}\nabla\mathbf{u}|^{\frac{3}{2}}\,d\mathcal{H}^2_{\mathbf{x}}, \fint_{S_R} (\det\nabla\mathbf{u})\,d\mathcal{H}^2_{\mathbf{x}}\right) \\
&\qquad\geq \Phi\left(\fint_{S_R} |\nabla\widetilde{\mathbf{u}}^{\mathrm{rad}}|^3\,d\mathcal{H}^2_{\mathbf{x}}, \fint_{S_R} |\operatorname{adj}\nabla\widetilde{\mathbf{u}}^{\mathrm{rad}}|^{\frac{3}{2}}\,d\mathcal{H}^2_{\mathbf{x}}, \fint_{S_R} (\det\nabla\widetilde{\mathbf{u}}^{\mathrm{rad}})\,d\mathcal{H}^2_{\mathbf{x}}\right)
\end{aligned}$$

and hence, if we multiply by $4\pi R^2$, integrate over $K \subset [a,b]$, and note that $|\nabla\widetilde{\mathbf{u}}^{\rm rad}|$, $|\operatorname{adj}\nabla\widetilde{\mathbf{u}}^{\rm rad}|$, and $\det\nabla\widetilde{\mathbf{u}}^{\rm rad}$ are constant on the sphere S_R, we find that

$$\int_K\int_{S_R}\Phi\big(|\nabla\mathbf{u}|^3, |\operatorname{adj}\nabla\mathbf{u}|^{\frac{3}{2}}, \det\nabla\mathbf{u}\big)\,d\mathcal{H}^2_{\mathbf{x}}\,dR \\ \geq \int_K\int_{S_R}\Phi\big(|\nabla\widetilde{\mathbf{u}}^{\rm rad}|^3, |\operatorname{adj}\nabla\widetilde{\mathbf{u}}^{\rm rad}|^{\frac{3}{2}}, (\det\nabla\widetilde{\mathbf{u}}^{\rm rad})\big)\,d\mathcal{H}^2_{\mathbf{x}}\,dR. \quad (5.16)$$

Next consider $R \in (a_k, b_k)$. Define A_k to be the annulus $\{\mathbf{x} \in \mathbb{R}^3 : a_k < |\mathbf{x}| < b_k\}$. Then in view of (5.6)–(5.8), Jensen's inequality and the monotonicity of Φ in its first two arguments yield

$$⨍_{A_k}\Phi(|\nabla\mathbf{u}|^3, |\operatorname{adj}\nabla\mathbf{u}|^{\frac{3}{2}}, \det\nabla\mathbf{u})\,d\mathbf{x} \\ \geq \Phi\Big(⨍_{A_k}|\nabla\mathbf{u}|^3\,d\mathbf{x}, ⨍_{A_k}|\operatorname{adj}\nabla\mathbf{u}|^{\frac{3}{2}}\,d\mathbf{x}, ⨍_{A_k}(\det\nabla\mathbf{u})\,d\mathbf{x}\Big) \\ \geq \Phi\Big(⨍_{A_k}|\nabla\widetilde{\mathbf{u}}^{\rm rad}|^3\,d\mathbf{x}, ⨍_{A_k}|\operatorname{adj}\nabla\widetilde{\mathbf{u}}^{\rm rad}|^{\frac{3}{2}}\,d\mathbf{x}, ⨍_{A_k}(\det\nabla\widetilde{\mathbf{u}}^{\rm rad})\,d\mathbf{x}\Big)$$

and hence, if we multiply by the volume of A_k and note that, by Proposition 5.9, $|\nabla\widetilde{\mathbf{u}}^{\rm rad}|$, $|\operatorname{adj}\nabla\widetilde{\mathbf{u}}^{\rm rad}|$, and $\det\nabla\widetilde{\mathbf{u}}^{\rm rad}$ are constant on the annulus A_k, we find that

$$\int_{A_k}\Phi\big(|\nabla\mathbf{u}|^3, |\operatorname{adj}\nabla\mathbf{u}|^{\frac{3}{2}}, \det\nabla\mathbf{u}\big)\,d\mathbf{x} \geq \int_{A_k}\Phi\big(|\nabla\widetilde{\mathbf{u}}^{\rm rad}|^3, |\operatorname{adj}\nabla\widetilde{\mathbf{u}}^{\rm rad}|^{\frac{3}{2}}, (\det\nabla\widetilde{\mathbf{u}}^{\rm rad})\big)\,d\mathbf{x},$$

which, together with (5.16) and the dominated convergence theorem, implies the desired result. □

Remark 5.13. The stored-energy functions (5.11) are by no means the only classes of stored-energy functions for which symmetrisation lowers the energy. For example, direct computations show that symmetrisation lowers the total energy for the stored-energy function $W(\mathbf{F}) = |\mathbf{F}||\operatorname{adj}\mathbf{F}|$ and also for $W(\mathbf{F}) = |\mathbf{F}|^q|\operatorname{adj}\mathbf{F}|^2$ for any $q \in (0,4)$.

Remark 5.14. The energy of $\mathbf{u}$ will be *strictly greater* than the energy of $\widetilde{\mathbf{u}}^{\rm rad}$ if any of the inequalities used in our derivation is strict. In particular, in equation (3.4) (and (3.19))

$$\Big|(\nabla\mathbf{u})\mathbf{n}\cdot(\operatorname{adj}\nabla\mathbf{u})^{\rm T}\mathbf{n}\Big| \leq \Big|(\nabla\mathbf{u})\mathbf{n}\Big|\,\Big|(\operatorname{adj}\nabla\mathbf{u})^{\rm T}\mathbf{n}\Big|$$

is a strict inequality unless the vectors $(\nabla\mathbf{u})\mathbf{n}$ and $(\operatorname{adj}\nabla\mathbf{u})^{\rm T}\mathbf{n}$ are parallel; in which case it then follows that $\mathbf{n}$ is an eigenvector of $(\nabla\mathbf{u})^{\rm T}\nabla\mathbf{u}$.

Remark 5.15. The corresponding result in the two-dimensional case, $n = 2$, is that if

$$W(\mathbf{F}) = \Phi(|\mathbf{F}|^2, \det \mathbf{F}),$$

where $\Phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is convex and $s \mapsto \Phi(s, j)$ is monotone increasing, then $E(\mathbf{u}) \geq E(\widetilde{\mathbf{u}}^{\mathrm{rad}})$ for any $\mathbf{u} \in \mathcal{A}_\mu^\lambda$. The proof of this follows exactly as in Theorem 5.12 on noting the results of Remark 3.7.

Remark 5.16. A similar result is valid when $n \geq 4$. A slight modification of our proof will show that if the energy satisfies

$$W(\mathbf{F}) = \Phi(|\mathbf{F}|^n, |\operatorname{adj} \mathbf{F}|^{\frac{n}{n-1}}, \det \mathbf{F}),$$

where $\Phi : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is convex and monotone increasing in its first two arguments, then $E(\mathbf{u}) \geq E(\widetilde{\mathbf{u}}^{\mathrm{rad}})$ for any $\mathbf{u} \in \mathcal{A}_\mu^\lambda$.

Finally, we briefly consider the mixed problem. Without the additional displacement boundary condition on the inner boundary it is clear that there are deformations $\mathbf{u} \in \mathcal{A}_\lambda^O$ whose radial symmetrisation does not satisfy $\mu \geq \lambda$ (see (5.4)). Thus our techniques will not yield a comparison of the energy of such a deformation with the energy of its radial symmetrisation. However, if the global minimiser of the energy among radial deformations exists and satisfies $\mu > \lambda$ then our results do yield such a comparison for all $\mathbf{u} \in \mathcal{A}_\lambda^O$ that lie in a neighbourhood (in the C^0-topology) of the radial minimiser. This establishes the following result.

Corollary 5.17. *Let $\lambda > 0$ and suppose that $W(\mathbf{F}) = \Phi(|\mathbf{F}|^3, |\operatorname{adj} \mathbf{F}|^{\frac{3}{2}}, \det \mathbf{F})$ satisfies the hypotheses of Theorem 5.12. Suppose further that there exists a radial deformation*

$$\mathbf{w}^{\mathrm{r}}(\mathbf{x}) = \frac{r(R)}{R}\mathbf{x} \in \mathcal{A}_\lambda^O \cap \mathcal{R}_\lambda^O$$

that minimises the elastic energy

$$E(\mathbf{z}^{\mathrm{r}}) = \int_A W(\nabla \mathbf{z}^{\mathrm{r}}(\mathbf{x}))\, d\mathbf{x}$$

amongst all $\mathbf{z}^{\mathrm{r}} \in \mathcal{R}_\lambda^O$. Define $\mu := r(a)/a$ and suppose that $\mu > \lambda$. Then $\mathbf{w}^{\mathrm{r}}$ is a strong relative minimiser of the energy, that is, there is a neighbourhood of $\mathbf{w}^{\mathrm{r}}$ in $\mathcal{L}^\infty(A; \mathbb{R}^3)$ such that any $\mathbf{w} \in \mathcal{A}_\lambda^O$ that lies in this neighbourhood satisfies $E(\mathbf{w}) \geq E(\mathbf{w}^{\mathrm{r}})$.

Remark 5.18. To apply the above Corollary, we refer to results in [4], [17], and [10] which can be adapted to prove that there exists an energy minimiser $\mathbf{w}^{\mathrm{r}}$ in the class of radial deformations $\mathcal{R}_\lambda^O$.

Example 5.19. Suppose that $p \geq 3$ and that $h \in C^2(\mathbb{R}^+; [0, \infty))$ is convex with $h(d) \to \infty$ as $d \to 0^+$. Then the stored-energy function

$$W(\mathbf{F}) = |\mathbf{F}|^p + h(\det \mathbf{F})$$

clearly satisfies the hypotheses of Theorem 5.12. In addition W is globally strongly-elliptic and hence it follows from results in [4, 17] that, for each positive $\lambda \neq 1$, the mixed problem has a unique radial minimiser that satisfies either

$$\frac{r(R)}{R} > r'(R) \ \text{ for every } R \in [a, b] \quad \text{or} \quad \frac{r(R)}{R} < r'(R) \ \text{ for every } R \in [a, b].$$

For this particular stored-energy function, it follows from phase-plane arguments for the radial equilibrium equation (see [17, section 2]), that the first of the above inequalities holds in the case $\lambda > 1$ and the second in the case $\lambda < 1$.

Remark 5.20. Let $\mathbf{N}$ denote the outward unit normal to $\mathbf{u}(A)$. Then, for any $P \in \mathbb{R}$, Corollary 5.17 is also valid for the energy

$$\mathbb{E}_P(\mathbf{u}) := \int_A W(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x} - P \int_{\mathbf{u}(S_a)} \mathbf{y} \cdot \mathbf{N}(\mathbf{y})\, d\mathcal{H}^2_{\mathbf{y}},$$

which loads the (deformed) inner boundary with a constant pressure. This follows from the fact that our summarisations preserve volume and the integral that multiplies P is three times the volume of the region contained within the surface $\mathbf{u}(S_a)$ (cf. the concluding remarks in [18]).

6 Symmetry of Energy Minimising Deformations when $\lambda > \mu$.

In this section we will show that the arguments used in the previous section with $\mu \geq \lambda$ can, for the pure displacement problem, be applied to certain classes of polyconvex stored-energy functions when $\lambda > \mu$. Initially, we again let $A \subset \mathbb{R}^n$ for some integer $n \geq 2$.

We first note that results from degree theory (see, e.g., [9] or [16]) imply that the image of A under any continuous, one-to-one map that satisfies the given boundary conditions is the annulus

$$A^* = \{\mathbf{x} \in \mathbb{R}^n : \mu a < |\mathbf{x}| < \lambda b\}, \tag{6.1}$$

and, moreover, that such a map is open and satisfies $\mathbf{u}(A) = A^*$. The following result[5] then follows from degree theory and the inverse function theorem.

[5] See, e.g., [6, Theorem 5.5-2] and recall that $\mathbf{u} \in C^1(\overline{A})$ means $\mathbf{u}$ is C^1 on a open set containing $\overline{A}$.

Proposition 6.1. *Let $0 < \mu a < \lambda b$ and $\mathbf{u} \in \mathcal{A}_\mu^\lambda$. Then $\mathbf{u}$ is one-to-one and satisfies $\mathbf{u}(A) = A^*$. Moreover, $\mathbf{u}$ has an inverse $\mathbf{v} = \mathbf{v}_\mathbf{u} := \mathbf{u}^{-1} \in C^1(\overline{A^*}; \overline{A})$; this inverse is one-to-one and satisfies $\mathbf{v}(A^*) = A$ and*

$$\mathbf{v}(\mathbf{y}) = \mu^{-1}\mathbf{y} \text{ for } \mathbf{y} \in \partial A_I^*, \qquad \mathbf{v}(\mathbf{y}) = \lambda^{-1}\mathbf{y} \text{ for } \mathbf{y} \in \partial A_o^*,$$

where ∂A_I^* and ∂A_o^* are the inner and outer boundaries of A^*, respectively.

We now follow Ball [2, pp. 210–211] and change variables to the deformed configuration to prove the main result of this section.

Theorem 6.2. *Let $\lambda > \mu$ and suppose that $\widehat{W}(\mathbf{F}) := (\det \mathbf{F})\, W(\mathbf{F}^{-1})$ satisfies the hypotheses of Theorem 5.12. Then for each $\mathbf{u} \in \mathcal{A}_\mu^\lambda$ there exists a bi-Lipschitz radial deformation $\widetilde{\mathbf{z}}^{\rm rad} \in \mathcal{R}_\mu^\lambda$ such that $E(\mathbf{u}) \geq E(\widetilde{\mathbf{z}}^{\rm rad})$.*

Proof. Let $\mathbf{u} \in \mathcal{A}_\mu^\lambda$ with inverse $\mathbf{v}$ so that $(\mathbf{u} \circ \mathbf{v})(\mathbf{y}) = \mathbf{y}$. Then, by the chain rule $[\nabla_\mathbf{x}\mathbf{u}(\mathbf{v}(\mathbf{y}))][\nabla_\mathbf{y}\mathbf{v}(\mathbf{y})] = \mathbf{I}$ and hence, for each $\mathbf{y} \in \overline{A^*}$,

$$\nabla_\mathbf{x}\mathbf{u}(\mathbf{v}(\mathbf{y})) = \Big[\nabla_\mathbf{y}\mathbf{v}(\mathbf{y})\Big]^{-1}. \tag{6.2}$$

The change of variables formula for multiple integrals together with (6.2), Proposition 6.1, and the definition of $\widehat{W}$ then imply

$$\begin{aligned}\int_A W(\nabla_\mathbf{x}\mathbf{u}(\mathbf{x}))\, d\mathbf{x} &= \int_{A^*} W\big(\nabla_\mathbf{x}\mathbf{u}(\mathbf{v}(\mathbf{y}))\big)\big(\det \nabla_\mathbf{y}\mathbf{v}(\mathbf{y})\big)\, d\mathbf{y} \\ &= \int_{A^*} \widehat{W}(\nabla_\mathbf{y}\mathbf{v}(\mathbf{y}))\, d\mathbf{y},\end{aligned} \tag{6.3}$$

where $A^* = \mathbf{u}(A)$ is given by (6.1).

Now apply Theorem 5.12 with A, W, and $\mathbf{u}$ replaced by A^*, $\widehat{W}$, and $\mathbf{v}$, respectively, noting that $\lambda^{-1} < \mu^{-1}$ and that $\mathbf{v}(\mathbf{y}) = \lambda^{-1}\mathbf{y}$ on ∂A_o^* and $\mathbf{v}(\mathbf{y}) = \mu^{-1}\mathbf{y}$ on ∂A_I^*. Thus, there exists a bi-Lipschitz radial map $\widetilde{\mathbf{v}}^{\rm rad} : \overline{A^*} \to \overline{A}$ that satisfies

$$\int_{A^*} \widehat{W}(\nabla\mathbf{v}(\mathbf{y}))\, d\mathbf{y} \geq \int_{A^*} \widehat{W}(\nabla\widetilde{\mathbf{v}}^{\rm rad}(\mathbf{y}))\, d\mathbf{y}. \tag{6.4}$$

We next note that since $\widetilde{\mathbf{v}}^{\rm rad}$ is radial and bi-Lipschitz so is its inverse $\widetilde{\mathbf{z}}^{\rm rad} : \overline{A} \to \overline{A^*}$. It follows that $\widetilde{\mathbf{z}}^{\rm rad} \in \mathcal{R}_\mu^\lambda$. Moreover, the change of variables formula, (6.3), as well as the identity (6.2) (for a.e. $\mathbf{y} \in A^*$) are also valid for Lipschitz functions.[6] The desired result now follows from (6.4) together with (6.2) and (6.3) with $\mathbf{u}$ and its inverse $\mathbf{v}$ replaced by $\widetilde{\mathbf{z}}^{\rm rad}$ and its inverse $\widetilde{\mathbf{v}}^{\rm rad}$, respectively. □

[6] See, e.g., [8, Theorem 3.2.5] and [7, Chapter 3].

The next proposition can be helpful in identifying stored-energy functions for which the last theorem applies.

Proposition 6.3. *Suppose that* $\Psi : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is continuous. Then for any* $p, q, s, t \in \mathbb{R}$*, the mapping* $\mathbf{u}$ *and its inverse* $\mathbf{v}$ *satisfy*

$$\int_A \Psi\left(\frac{|\nabla_{\mathbf{x}}\mathbf{u}|^s}{(\det \nabla_{\mathbf{x}}\mathbf{u})^p}, \frac{|\operatorname{adj} \nabla_{\mathbf{x}}\mathbf{u}|^t}{(\det \nabla_{\mathbf{x}}\mathbf{u})^q}, \det \nabla_{\mathbf{x}}\mathbf{u}\right) d\mathbf{x}$$
$$= \int_{A^*} \Psi\left(\frac{|\operatorname{adj} \nabla_{\mathbf{y}}\mathbf{v}|^s}{(\det \nabla_{\mathbf{y}}\mathbf{v})^{s-p}}, \frac{|\nabla_{\mathbf{y}}\mathbf{v}|^t}{(\det \nabla_{\mathbf{y}}\mathbf{v})^{t-q}}, [\det \nabla_{\mathbf{y}}\mathbf{v}]^{-1}\right)(\det \nabla_{\mathbf{y}}\mathbf{v})\, d\mathbf{y}.$$

Proof. Let $\mathbf{u} \in \mathcal{A}_\mu^\lambda$ with inverse $\mathbf{v}$. Then the change of variables formula yields

$$\int_A \Psi\left(\frac{|\nabla_{\mathbf{x}}\mathbf{u}|^s}{(\det \nabla_{\mathbf{x}}\mathbf{u})^p}, \frac{|\operatorname{adj} \nabla_{\mathbf{x}}\mathbf{u}|^t}{(\det \nabla_{\mathbf{x}}\mathbf{u})^q}, \det \nabla_{\mathbf{x}}\mathbf{u}\right) d\mathbf{x}$$
$$= \int_{A^*} \Psi\left(\frac{|\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{v}(\mathbf{y}))|^s}{(\det \nabla_{\mathbf{x}}\mathbf{u}(\mathbf{v}(\mathbf{y})))^p}, \frac{|\operatorname{adj} \nabla_{\mathbf{x}}\mathbf{u}(\mathbf{v}(\mathbf{y}))|^t}{(\det \nabla_{\mathbf{x}}\mathbf{u}(\mathbf{v}(\mathbf{y})))^q}, \det \nabla_{\mathbf{x}}\mathbf{u}(\mathbf{v}(\mathbf{y}))\right)(\det \nabla_{\mathbf{y}}\mathbf{v}(\mathbf{y}))\, d\mathbf{y}. \quad (6.5)$$

Next, (6.2) yields $\nabla_{\mathbf{x}}\mathbf{u} = [\nabla_{\mathbf{y}}\mathbf{v}]^{-1}$ and hence

$$\det \nabla_{\mathbf{x}}\mathbf{u}(\mathbf{v}(\mathbf{y})) = [\det \nabla_{\mathbf{y}}\mathbf{v}(\mathbf{y})]^{-1}.$$

Consequently, since $\operatorname{adj}\mathbf{F} = (\det \mathbf{F})\mathbf{F}^{-1}$ and $\operatorname{adj}(\operatorname{adj}\mathbf{F}) = (\det \mathbf{F})\mathbf{F}$ (cf. (3.18))

$$\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{v}(\mathbf{y})) = \operatorname{adj}\big[\nabla_{\mathbf{y}}\mathbf{v}(\mathbf{y})\big]\,[\det \nabla_{\mathbf{y}}\mathbf{v}(\mathbf{y})]^{-1}$$

$$\operatorname{adj}[\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{v}(\mathbf{y}))] = \nabla_{\mathbf{y}}\mathbf{v}(\mathbf{y})\,[\det \nabla_{\mathbf{y}}\mathbf{v}(\mathbf{y})]^{-1}.$$

The last three equations and (6.5) then yield the desired result. □

Example 6.4. Let $\mu < \lambda$ and

$$W(\mathbf{F}) = \Psi\left(\frac{|\mathbf{F}|^{3/2}}{(\det \mathbf{F})^{1/2}}, \frac{|\operatorname{adj}\mathbf{F}|^3}{(\det \mathbf{F})^2}, \det \mathbf{F}\right),$$

where $\Psi : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is monotone increasing in its first two arguments and convex. Then, for any $\mathbf{u} \in \mathcal{A}_\mu^\lambda$, there is a radial deformation $\widetilde{\mathbf{z}}^{\mathrm{rad}} \in \mathcal{R}_\mu^\lambda$ that satisfies $E(\mathbf{u}) \geq E(\widetilde{\mathbf{z}}^{\mathrm{rad}})$.

Proof. This follows from Theorem 6.2 on noting that

$$\widehat{W}(\mathbf{F}) = (\det \mathbf{F})\, W(\mathbf{F}^{-1}) = (\det \mathbf{F})\, G\left(\frac{|\operatorname{adj}\mathbf{F}|^{\frac{3}{2}}}{\det \mathbf{F}}, \frac{|\mathbf{F}|^3}{\det \mathbf{F}}, \frac{1}{\det \mathbf{F}}\right)$$

which is easily shown to be jointly convex in the variables[7] $(|\operatorname{adj}\mathbf{F}|^{\frac{3}{2}}, |\mathbf{F}|^3, \det \mathbf{F})$. □

[7] Note that $(q, s, t) \mapsto G(q, s, t)$ convex implies $(q, s, t) \mapsto t\, G\left(\frac{q}{t}, \frac{s}{t}, \frac{1}{t}\right)$ is also convex.

Remark 6.5. As noted for the incompressible case in Part I [18], the results in previous sections rely on the idea that the image of each sphere centred at the origin in the reference configuration, $S_R \subset A$, prefers to retain its spherical shape in order to minimise the elastic energy. The results in the current section instead use the property that the preimage of any sphere centred at the origin, in the deformed configuration, $S_r \subset A^*$, prefers to be the image of some sphere centred at the origin. This idea cannot be applied unless the deformed configuration is the union of such spheres, which necessitates that the image of the annulus A be another annulus. Hence the technique in this section is only applicable to the pure displacement problem.

7 Sobolev Deformations.

We now generalise the results of the previous sections to allow for deformations given by the existence theory of Ball [1] (and subsequent generalisation in [5, 14, 19, 20]). For this entire section we again take $A \subset \mathbb{R}^n$ with $n = 2$ or $n = 3$.

Definition 7.1. Suppose that $p > n$ and $0 < \mu a < \lambda b$. For the pure displacement problem we define the set of admissible *Sobolev deformations* by

$$\mathcal{S}^p = \left\{ \mathbf{u} \in W^{1,p}(A;\mathbb{R}^n) \cap C^0(\overline{A};\mathbb{R}^n) : \det \nabla \mathbf{u} > 0 \ a.e., \quad \begin{array}{l} \mathbf{u}(\mathbf{x}) = \mu \mathbf{x} \text{ for } \mathbf{x} \in \partial A_I, \\ \mathbf{u}(\mathbf{x}) = \lambda \mathbf{x} \text{ for } \mathbf{x} \in \partial A_o \end{array} \right\},$$

where $W^{1,p}(A;\mathbb{R}^n)$ denotes the usual Sobolev space of vector-valued functions $\mathbf{u} \in \mathcal{L}^p(A;\mathbb{R}^n)$, whose distributional derivative also lies in $\mathcal{L}^p$.

The next result, which gives invertibility properties of deformations, is due to Ball [3].

Proposition 7.2. *Let $p > n$ and $0 < \mu a < \lambda b$. Suppose that $\mathbf{u} \in \mathcal{S}^p$. Then*

(a) There exists a Lebesgue null set $\mathcal{N} \subset A$ such that $\mathbf{u}$ is one-to-one on $A \backslash \mathcal{N}$; and

(b) $\mathbf{u}(\overline{A}) = \overline{A^*}$.

Suppose further that $|\operatorname{adj} \nabla \mathbf{u}|^q (\det \nabla \mathbf{u})^{1-q} \in \mathcal{L}^1(A)$ for some $q > n$, then

(c) $\mathbf{u}$ is one-to-one on $\overline{A}$;

(d) $\mathbf{u}(A) = A^*$;

(e) $\mathbf{u}$ has an inverse $\mathbf{v} \in W^{1,q}(A^*) \cap C^0(\overline{A^*})$ that satisfies $\nabla_{\mathbf{y}} \mathbf{v}(\mathbf{y}) = [\nabla_{\mathbf{x}} \mathbf{u}(\mathbf{v}(\mathbf{y}))]^{-1}$ for a.e. $\mathbf{y} \in A^*$, where $\nabla \mathbf{w}$ denotes the matrix of weak derivatives of a mapping $\mathbf{w}$; and

(f) $\det \nabla \mathbf{v}(\mathbf{y}) > 0$ for a.e. $\mathbf{y} \in A^*$.

Before proceeding further we note certain other key properties of such mappings.

Proposition 7.3. *Let $p > n$ and suppose that $\mathbf{u} \in W^{1,p}(A;\mathbb{R}^n) \cap C^0(\overline{A};\mathbb{R}^n)$ is one-to-one a.e. and satisfies* $\det \nabla \mathbf{u} > 0$ *a.e. on A. Then for $\mathcal{L}^1$ a.e. $R \in (a,b)$,*

(a) $\mathbf{u}|_{S_R} \in W^{1,p}(S_R) \cap C^0(S_R)$*;*

(b) $\mathbf{u}(S_R)$ *is $\mathcal{H}^{n-1}$ measurable with* $\mathcal{H}^{n-1}(\mathbf{u}(S_R)) \leq \int_{S_R} |(\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}} \mathbf{n}| \, d\mathcal{H}^{n-1}_{\mathbf{x}}$*;*

(c) $\partial^* (\operatorname{im}_T(\mathbf{u}, S_R))$ *is $\mathcal{H}^{n-1}$ measurable with* $\mathcal{H}^{n-1}(\partial^*(\operatorname{im}_{\mathrm{T}}(\mathbf{u}, S_R))) = \mathcal{H}^{n-1}(\mathbf{u}(S_R))$*;*

(d) $n^n \omega_n [\mathcal{L}^n(\operatorname{im}_{\mathrm{T}}(\mathbf{u}, S_R))]^{n-1} \leq \left[\mathcal{H}^{n-1}(\partial^*(\operatorname{im}_{\mathrm{T}}(\mathbf{u}, S_R)))\right]^n$*;*

(e) For any $\mathbf{v} \in C^1(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \operatorname{degree}(\mathbf{u}, S_R, \mathbf{y}) \operatorname{div} \mathbf{v}(\mathbf{y}) \, d\mathbf{y} = \int_{S_R} \mathbf{v}(\mathbf{u}(\mathbf{x})) \cdot (\operatorname{adj} \nabla \mathbf{u}(\mathbf{x}))^{\mathrm{T}} \mathbf{n}(\mathbf{x}) \, d\mathcal{H}^{n-1}_{\mathbf{x}}; \textit{and}$$

(f) Each $\mathbf{y} \in \mathbb{R}^n \backslash \mathbf{u}(S_R)$ *satisfies* $\operatorname{degree}(\mathbf{u}, S_R, \mathbf{y}) = 1$ *or* $\operatorname{degree}(\mathbf{u}, S_R, \mathbf{y}) = 0$.

Here

$$\operatorname{im}_{\mathrm{T}}(\mathbf{u}, S_R) := \{\mathbf{y} \in \mathbb{R}^n \backslash \mathbf{u}(S_R) : \operatorname{degree}(\mathbf{u}, S_R, \mathbf{y}) \neq 0\} \tag{7.1}$$

is the topological image of S_R under $\mathbf{u}$, $\mathcal{H}^{n-1}$ denotes $(n-1)$-dimensional Hausdorff measure, $\partial^*\Omega$ denotes the reduced boundary[8] *of Ω, $\mathcal{L}^n$ denotes n-dimensional Lebesgue measure, ω_n is the volume of the unit ball in $\mathbb{R}^n$ ($\omega_2 = \pi$ and $\omega_3 = 4\pi/3$) and* degree *denotes the Brouwer degree.*[9]

Remark 7.4. Proposition 7.3(a) is well-known, see, e.g., [7, 9, 21]. Part (b) is due to Marcus and Mizel [11] (see also [8, 9]). Part (c) can be found in, e.g., the proof of Lemma 3.5 (steps 1–3) in [12]. Part (d) is a version of the classical isoperimetric inequality. It can be found in, for example, [7, p. 190] or [21, Theorem 5.4.3]; the given (dimensionally dependent) constant $n^n\omega_n$ can be found in Federer [8, pp. 275, 278]. Part (e) can be found in, e.g., [13, Proposition 2.1].

Proof of (f). Let $p > n$ and $\mathbf{u} \in \mathcal{S}^p$. Define an extension of $\mathbf{u}$ by

$$\mathbf{u}^e(\mathbf{x}) = \begin{cases} \mathbf{u}(\mathbf{x}), & \text{if } \mathbf{x} \in \overline{A}, \\ \mu \mathbf{x}, & \text{if } \mathbf{x} \in B_a = B_b \backslash \overline{A}. \end{cases}$$

Clearly, $\mathbf{u}^e \in W^{1,p}(B_b) \cap C^0(\overline{B}_b)$ and $\det \nabla \mathbf{u}^e > 0$ a.e. in B_b.

Next, in view of Proposition 7.2, $\mathbf{u}^e$ is one-to-one a.e. on B_b. Consequently, $\mathbf{u}^e$ satisfies condition (INV) of Müller and Spector [12] and hence we can apply Lemma 3.5 in [12] to conclude that the degree only assumes the values zero and one. Finally, for $R \in (a,b)$ the functions $\mathbf{u}$ and $\mathbf{u}^e$ are equal, which establishes (f) as the degree only depends on the boundary values. □

[8] See, e.g., Chapter 5 in either [7] or [21].

[9] See, e.g., [9] or [16].

7.1 Symmetrisation of Sobolev Deformations.

We first prove the analogue of Proposition 2.3 for Sobolev deformations $\mathbf{u} \in \mathcal{S}^p$. For each such $\mathbf{u}$ we define its radial symmetrisation

$$\mathbf{u}^{\mathrm{rad}}(\mathbf{x}) = \frac{r(R)}{R}\mathbf{x}, \quad \omega_n r(R)^n := \omega_n \lambda^n b^n - \int_{B_b \setminus B_R} (\det \nabla \mathbf{u})\, d\mathbf{x}. \tag{7.2}$$

Lemma 7.5. *Let $\lambda > 0$, $p > n$, and $\mathbf{u} \in \mathcal{S}^p$. Then for almost every $R \in (a, b)$*

$$\mathcal{H}^{n-1}(\mathbf{u}(S_R)) \geq \mathcal{H}^{n-1}(\mathbf{u}^{\mathrm{rad}}(S_R)),$$

where $\mathbf{u}^{\mathrm{rad}}$ is given by (7.2).

Proof. Fix $R \in [a, b]$. Let $\mathbf{u} \in C^2(\overline{A}; \mathbb{R}^n)$ satisfy $\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x}$ for $\mathbf{x} \in S_b$. Then the well-known divergence identity

$$\det \nabla \mathbf{u} = \frac{\partial}{\partial x^\alpha}\left(\frac{1}{n}\mathrm{u}^i (\operatorname{adj} \nabla \mathbf{u})_i^\alpha\right) = \operatorname{div}\left(\frac{1}{n}(\operatorname{adj} \nabla \mathbf{u})\mathbf{u}\right),$$

when integrated over $B_b \setminus B_R$, yields

$$\int_{B_b \setminus B_R} (\det \nabla \mathbf{u})\, d\mathbf{x} = \omega_n \lambda^n b^n - \int_{S_R} \frac{1}{n}\mathbf{u} \cdot (\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}} \mathbf{n}\, d\mathcal{H}^{n-1}_{\mathbf{x}}, \tag{7.3}$$

where $\mathbf{n}$ is the outward unit normal to $B_b \setminus B_R$. However, the boundary is smooth and so $C^2(\overline{A}; \mathbb{R}^n)$ is dense in $W^{1,p}(A; \mathbb{R}^n)$. Now let $\mathbf{u} \in W^{1,p}(A; \mathbb{R}^n)$ satisfy $\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x}$ and consider a sequence of C^2 mappings $\mathbf{u}_n \to \mathbf{u}$ strongly in $W^{1,p}(A; \mathbb{R}^n)$. Then, $p > n$ yields $\mathbf{u}_n \to \mathbf{u}$ uniformly on $\overline{A}$, by the Sobolev imbedding theorem, and $\det \nabla \mathbf{u}_n \to \det \nabla \mathbf{u}$ strongly in $\mathcal{L}^1(A)$. Next, by Proposition 7.3(a) together with Fubini's theorem, for $\mathcal{L}^1$ a.e. $R \in (a, b)$, there is a subsequence (not relabelled) that satisfies $\mathbf{u}_n \to \mathbf{u}$ strongly in $W^{1,p}(S_R; \mathbb{R}^n)$ and consequently $\operatorname{adj} \nabla \mathbf{u}_n \to \operatorname{adj} \nabla \mathbf{u}$ strongly in $\mathcal{L}^1(S_R; \mathrm{M}^{n\times n})$. We therefore conclude that, for a.e. $R \in (a, b)$, (7.3) is satisfied by all $\mathbf{u} \in W^{1,p}(A; \mathbb{R}^n)$ that obey $\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x}$ on S_b and, in particular, for all $\mathbf{u} \in \mathcal{S}^p$.

We next note that, since $\mathbf{u}^{\mathrm{rad}}(S_R)$ is the sphere of radius $r(R)$ given by (7.2),

$$n^n \omega_n \Big[\mathcal{L}^n\left(B_{r(R)}\right)\Big]^{n-1} = \Big[\mathcal{H}^n\left(\mathbf{u}(S_R)\right)\Big]^n. \tag{7.4}$$

The desired result will follow from (7.4) and the isoperimetric inequality, Proposition 7.3(c,d), if we can establish that the open set $\mathrm{im}_{\mathrm{T}}(\mathbf{u}, S_R)$ has the same volume as $B_{r(R)}$, i.e., in view of (7.2) and (7.3),

$$\begin{aligned} \omega_n r(R)^n &= \omega_n \lambda^n b^n - \int_{B_b \setminus B_R} (\det \nabla \mathbf{u})\, d\mathbf{x} \\ &= \frac{1}{n}\int_{S_R} \mathbf{u} \cdot (\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}} \mathbf{n}\, d\mathcal{H}^{n-1}_{\mathbf{x}}. \end{aligned} \tag{7.5}$$

However, by (7.1) and Proposition 7.3(e,f), with $\mathbf{v}(\mathbf{x}) = \mathbf{x}/3$,

$$\mathcal{L}^n\left(\mathrm{im}_{\mathrm{T}}(\mathbf{u}, S_R)\right) = \frac{1}{n}\int_{S_R} \mathbf{u}\cdot(\mathrm{adj}\,\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\, d\mathcal{H}^{n-1}_{\mathbf{x}},$$

which together with (7.5) completes the proof. □

Remark 7.6. As is clear from the proof, the last result is independent of the regularity (or lack of regularity) of $\mathbf{u}^{\mathrm{rad}}$.

7.2 Regularity of Radial Symmetrisations.

In addition to the radial symmetrisation $\mathbf{u}^{\mathrm{rad}}$, in Proposition 5.9 we introduced a modified radial symmetrisation $\widetilde{\mathbf{u}}^{\mathrm{rad}}$ for smooth deformations $\mathbf{u}$. We now correspondingly define this *modified radial symmetrisation* for each $\mathbf{u} \in \mathcal{S}^p$, $p > n$, by

$$\widetilde{\mathbf{u}}^{\mathrm{rad}}(\mathbf{x}) = g(R)\,\mathbf{x}, \qquad g(R) = \max\left\{\lambda, \inf_{\tau\in[a,R]} \frac{r(\tau)}{\tau}\right\}. \tag{7.6}$$

In this section we investigate regularity properties of the symmetrisations $\mathbf{u}^{\mathrm{rad}}$ and $\widetilde{\mathbf{u}}^{\mathrm{rad}}$ of Sobolev maps.

First note that if $\mathbf{u} \in \mathcal{S}^p$, $p > n$, then $\det\nabla\mathbf{u} \in \mathcal{L}^{p/n}(A)$ with $\det\nabla\mathbf{u} > 0$ a.e. Now define corresponding maps $\psi, \psi^* : [a,b] \to \mathbb{R}$ by

$$\psi(R) := \int_{B_b\setminus B_R} (\det\nabla\mathbf{u})\, d\mathbf{x}, \qquad \psi^*(R) := \int_{S_R} (\det\nabla\mathbf{u})\, d\mathcal{H}^{n-1}_{\mathbf{x}}. \tag{7.7}$$

Then ψ is strictly decreasing, absolutely continuous on $[a,b]$, and satisfies $\psi(b) = 0$ and $\psi(a) = \lambda^n b^n - \mu^n a^n > 0$. Fubini's theorem then implies that ψ^* is also well-defined, finite almost everywhere, and satisfies

$$\psi(t) = \int_t^b \psi^*(R)\, dR.$$

Thus, $\psi'(R) = -\psi^*(R)$ for a.e. $R \in (a,b)$, where the prime here denotes the classical derivative. Moreover, since $\det\nabla\mathbf{u} > 0$ a.e. in A it follows that, for $\mathcal{L}^1$ almost every $R \in (a,b)$,

$$(\det\nabla\mathbf{u})\Big|_{S_R} > 0 \;\; \mathcal{H}^{n-1} \text{ a.e. on } S_R.$$

Consequently, $(7.7)_2$ yields $\psi'(R) = -\psi^*(R) < 0$ for almost every $R \in (a,b)$ and hence, using Hölder's inequality, $\psi' \in \mathcal{L}^1((a,b))$.

We now apply Lemma 5.6 together with the above observations to conclude the following result.

Proposition 7.7. *Let* $\mathbf{u} \in \mathcal{S}^p$ *with* $p > n$ *and define* $\mathbf{u}^{\rm rad} : A \to \mathbb{R}^n$ *by* (7.2). *Then* r *is absolutely continuous on* $[a, b]$ *with (classical) derivative* $r' \in L^1((a, b))$ *that satisfies* $r' > 0$ *a.e. Moreover,* $\mathbf{u}^{\rm rad} \in W^{1,1}(A; \mathbb{R}^n)$, *with distributional derivative given by* (2.2), *is one-to-one and satisfies* $\det \nabla \mathbf{u}^{\rm rad} > 0$ *a.e.*

The next result concerns the regularity of the modified radial symmetrisation $\widetilde{\mathbf{u}}^{\rm rad}$ defined by (7.6).

Corollary 7.8. *Let* $\mathbf{u} \in \mathcal{S}^p$ *with* $p > n$ *and suppose that* $\mathbf{u}^{\rm rad} \in W^{1,1}(A; \mathbb{R}^n)$ *is the radial symmetrisation of* $\mathbf{u}$ *given by* (7.2). *Assume that* $\mu > \lambda$. *Then the radial deformation* $\widetilde{\mathbf{u}}^{\rm rad}$ *given by* (7.6) *satisfies* $\widetilde{\mathbf{u}}^{\rm rad} \in W^{1,1}(A; \mathbb{R}^n)$ *with distributional derivative given by* (2.2). *Moreover,* $\widetilde{\mathbf{u}}^{\rm rad}$ *is one-to-one with* $\det \nabla \widetilde{\mathbf{u}}^{\rm rad} > 0$ *a.e.*

Proof. Let $\widetilde{\mathbf{u}}^{\rm rad}$ be the radial deformation defined by (7.6). By the previous proposition, r is absolutely continuous on $[a, b]$ with $a > 0$ and hence so is $f(\tau) := r(\tau)/\tau$. Then, in view of Lemma 5.1, g is absolutely continuous and monotone decreasing on $[a, b]$. Moreover, there are pairwise-disjoint open intervals $(a_k, b_k) \subset (a, b)$ such that $g \equiv g_k$ on (a_k, b_k); $g = f$ on $K := (a, b) \setminus \cup_k (a_k, b_k)$; and $g' = f' \chi_K$ a.e. In particular we note for future reference that, since g is monotone decreasing with $g(a) = \mu$ and $g(b) = \lambda$,

$$\lambda \le g_k \le \mu \text{ for all } k \in \mathbb{Z}^+. \tag{7.8}$$

Next, since g is absolutely continuous on $[a, b]$, so is $\rho(R) := Rg(R)$. In particular, ρ is differentiable a.e. with (classical) derivative $\rho' = Rg' + g$; thus, for a.e. $R \in (a, b)$,

$$\rho'(R) = \begin{cases} g_k & \text{for } R \in (a_k, b_k), \\ r'(R) & \text{for } R \in K. \end{cases} \tag{7.9}$$

Now $r' > 0$ a.e. and hence (7.8) and (7.9) imply

$$0 < \rho'(R) \le \max\{r'(R), \mu\} \text{ for a.e. } R \in (a, b).$$

We can now apply the previous proposition to conclude that $\rho' \in \mathcal{L}^1((a, b))$. Finally, Lemma 5.6 yields $\widetilde{\mathbf{u}}^{\rm rad} \in W^{1,1}(A; \mathbb{R}^n)$ with distributional derivative given by (2.2) and, since ρ is continuous with $\rho' > 0$ a.e., $\widetilde{\mathbf{u}}^{\rm rad}$ is one-to-one with $\det \nabla \widetilde{\mathbf{u}}^{\rm rad} > 0$ a.e. □

8 Energy Reduction for Sobolev Deformations.

In this section we again take $n = 3$ and extend the results in §3–§6 to Sobolev deformations.

8.1 The energy $W(\mathbf{F}) = \Phi(|\mathbf{F}|^3, |\operatorname{adj}\mathbf{F}|^{\frac{3}{2}}, \det\mathbf{F})$, revisited: $\mu \geq \lambda$.

The next theorem extends the results in §5.3 to Sobolev deformations $\mathbf{u} \in \mathcal{S}^p$.

Theorem 8.1. *Let* $0 < \mu a < \lambda b$ *and* $\mu \geq \lambda$. *Define*

$$E(\mathbf{u}) = \int_A W(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x},$$

where W *satisfies*

$$W(\mathbf{F}) = \Phi(|\mathbf{F}|^3, |\operatorname{adj}\mathbf{F}|^{\frac{3}{2}}, \det\mathbf{F})$$

with $\Phi : [0,\infty) \times [0,\infty) \times \mathbb{R}^+ \to \mathbb{R}$ *convex and* $s \mapsto \Phi(s,t,j)$ *and* $s \mapsto \Phi(t,s,j)$ *increasing functions for all* $t \geq 0$ *and* $j > 0$. *Then for each* $\mathbf{u} \in \mathcal{S}^p$, $p > 3$, *there is a radial map* $\widetilde{\mathbf{u}}^{\mathrm{rad}}$ *that satisfies*

$$E(\mathbf{u}) \geq E(\widetilde{\mathbf{u}}^{\mathrm{rad}}).$$

Assume further that $W(\mathbf{F}) \to \infty$ *as* $\det\mathbf{F} \to 0^+$ *and that there are constants* $\alpha > 1$, $c > 0$, *and* k *such that, for all* $s \geq 0$, $t \geq 0$, *and* $j > 0$,

$$\Phi(s,t,j) \geq cs^\alpha - k. \tag{8.1}$$

Then there exists a radial deformation $\mathbf{w}^{\mathrm{rad}} \in \mathcal{S}^p$, $p = 3\alpha$, *that is an absolute minimizer of the energy among those mappings contained in* $\mathcal{S}^p$.

Remark 8.2. If one additionally assumes that Φ is strictly convex, then a slight modification of results in Part I [18, Appendix] imply the radial minimiser $\mathbf{w}^{\mathrm{rad}} \in \mathcal{S}^p$ is the *unique* absolute minimiser of the energy. The main observations needed for this result are that if $E(\mathbf{u}) = E(\mathbf{w}^{\mathrm{rad}})$ then:

(i) $|\nabla\mathbf{u}|$, $|\operatorname{adj}\nabla\mathbf{u}|$, and $\det\nabla\mathbf{u}$ must be radial (i.e., functions of R only);

(ii) The vectors $(\nabla\mathbf{u})\mathbf{n}$ and $(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}$ must be parallel (see Remark 5.14); and

(iii) The isoperimetric inequality (see Lemma 7.5) is strict unless the image of each sphere is again a sphere.

With these observations one can prove that the deformed spheres $\mathbf{u}(S_R)$ must be centred at the origin and that $\mathbf{u}$ must be radial.

Proof of Theorem 8.1. If $\mu = \lambda$ then the polyconvexity (and hence quasiconvexity) of the integrand implies that $\mathbf{u}(\mathbf{x}) \equiv \lambda\mathbf{x}$ is an absolute minimiser of the energy. Thus we will assume that $\mu > \lambda$. Let $\mathbf{u} \in \mathcal{S}^p$, $p > 3$. If $E(\mathbf{u})$ is infinite then the result is clear. Thus we will further assume that $\mathbf{u}$ has finite energy.

Now, $\mathbf{u} \in \mathcal{S}^p$ implies $\mathbf{u} \in W^{1,p}(A;\mathbb{R}^3)$ and hence $|\operatorname{adj}\nabla\mathbf{u}| \in \mathcal{L}^{p/2}(A)$ and $\det\nabla\mathbf{u} \in \mathcal{L}^{p/3}(A)$. Thus, by Fubini's theorem, each of the integrals

$$\int_{S_R} |\nabla\mathbf{u}|^3\, d\mathcal{H}^2_{\mathbf{x}}, \quad \int_{S_R} |\operatorname{adj}\nabla\mathbf{u}|^{\frac{3}{2}}\, d\mathcal{H}^2_{\mathbf{x}}, \quad \int_{S_R} (\det\nabla\mathbf{u})\, d\mathcal{H}^2_{\mathbf{x}}, \tag{8.2}$$
$$\int_{S_R} \Phi(|\nabla\mathbf{u}|^3, |\operatorname{adj}\nabla\mathbf{u}|^{\frac{3}{2}}, \det\nabla\mathbf{u})\, d\mathcal{H}^2_{\mathbf{x}}$$

must be finite for $\mathcal{L}^1$ almost every $R \in (a,b)$. For such R, it then follows from $(8.2)_1$ that, in particular, $|(\nabla\mathbf{u}(\mathbf{x}))\mathbf{n}(\mathbf{x})|$ is finite for $\mathcal{H}^2$ a.e. $\mathbf{x} \in S_R$. In addition, $\mathbf{u} \in \mathcal{S}^p$ implies $\det\nabla\mathbf{u} > 0$ $\mathcal{L}^3$ a.e. and hence, for $\mathcal{L}^1$ a.e. $R \in (a,b)$,

$$\det\nabla\mathbf{u}(\mathbf{x}) > 0 \ \text{ for } \mathcal{H}^2 \ a.e. \ \mathbf{x} \in S_R.$$

Consequently, in view of (3.4), for $\mathcal{L}^1$ a.e. $R \in (a,b)$,

$$|(\operatorname{adj}\nabla\mathbf{u}(\mathbf{x}))^{\mathrm{T}}\mathbf{n}(\mathbf{x})| > 0 \text{ for } \mathcal{H}^2 \ a.e. \ \mathbf{x} \in S_R.$$

Therefore, for such R, previous arguments (see Lemma 3.2, Lemma 3.10, and their proofs) yield, for $\mathcal{H}^2$ a.e. $\mathbf{x} \in S_R$,

$$G\bigg(\det\nabla\mathbf{u}, \Big|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\Big|^{\frac{3}{2}}\bigg) \le |\nabla\mathbf{u}|^3, \qquad H\bigg(\det\nabla\mathbf{u}, \Big|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\Big|^{\frac{3}{2}}\bigg) \le |\operatorname{adj}\nabla\mathbf{u}|^{\frac{3}{2}},$$

where G and H are defined by (3.5) and (3.20), respectively.

We next make use of Jensen's inequality and the fact that each of the integrals in (8.2) is finite for a.e. R to conclude that

$$G\bigg(\fint_{S_R} \det\nabla\mathbf{u}, \fint_{S_R} \Big|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\Big|^{\frac{3}{2}}\bigg) \le \fint_{S_R} |\nabla\mathbf{u}|^3,$$
$$H\bigg(\fint_{S_R} \det\nabla\mathbf{u}, \fint_{S_R} \Big|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\Big|^{\frac{3}{2}}\bigg) \le \fint_{S_R} |\operatorname{adj}\nabla\mathbf{u}|^{\frac{3}{2}}$$

for $\mathcal{L}^1$ almost every $R \in (a,b)$.

In view of Proposition 7.7 and Corollary 7.8, define the symmetrisations $\mathbf{u}^{\mathrm{rad}}$ and $\widetilde{\mathbf{u}}^{\mathrm{rad}}$ by (7.2) and (7.6), respectively. Note that, by Lemma 5.1 and the proof of Corollary 7.8,

$$\begin{gathered} \rho(R) = g_k R, \quad \rho'(R) = g_k \ \text{ for } R \in (a_k, b_k), \\ \rho(R) = r(R), \quad \rho'(R) = r'(R) \ \text{ for a.e. } R \in K, \\ (a,b) = K \cup \bigcup_k (a_k, b_k), \qquad \lambda \le g_k \le \mu, \end{gathered} \tag{8.3}$$

where (a_k, b_k) are pairwise disjoint and $K \cap (a_k, b_k) = \varnothing$ for all k.

Now, the proof of Lemma 3.5 (with Proposition 2.3 replaced by Lemma 7.5) implies that (3.8)–(3.10) are satisfied for a.e. $R \in (a, b)$. Therefore, since $r'(R) \leq r(R)/R$ for a.e. $R \in K$, the computations used to prove Proposition 3.6 and Proposition 3.13 together with (2.5) and (8.3) now yield (5.13)–(5.15) for a.e. $R \in K$. The remainder of the proof is exactly analogous to the proof of Theorem 5.12 and allows us to conclude that the modified radial symmetrisation $\widetilde{\mathbf{u}}^{\mathrm{rad}}$ (defined by (7.6)) satisfies $E(\mathbf{u}) \geq E(\widetilde{\mathbf{u}}^{\mathrm{rad}})$.

Finally, the existence of a radial energy minimiser $\mathbf{w}^{\mathrm{rad}} \in \mathcal{S}^p$ follows by applying the symmetrisation procedure to an absolute minimiser $\mathbf{u}_m \in \mathcal{S}^p$ of E, whose existence is guaranteed by the results of Ball [1] under the further hypotheses we have made. See also [6, pp. 371–377]. □

8.2 The energy $W(\mathbf{F}) = \Phi(|\mathbf{F}|^{\frac{3}{2}}, |\operatorname{adj}\mathbf{F}|^3, \det \mathbf{F})$, revisited: $\lambda > \mu$.

The next theorem extends the results in §6 to Sobolev deformations.

Theorem 8.3. *Let $\lambda > \mu$ and suppose that $\widehat{W}(\mathbf{F}) := (\det \mathbf{F})\, W(\mathbf{F}^{-1})$ satisfies the hypotheses of Theorem 8.1. Suppose further that $\widehat{W}$ satisfies the (strengthened) growth condition*

$$\widehat{W}(\mathbf{F}) \geq c\left(|\mathbf{F}|^{\beta} + \frac{|\operatorname{adj}\mathbf{F}|^{\gamma}}{(\det \mathbf{F})^{\gamma-1}}\right) - k \quad \text{for all } \mathbf{F} \in \mathrm{M}^{n\times n}_{+},$$

for some constants $c > 0$, $\beta > 3$, $\gamma > 3$, and k. Then for each $\mathbf{u} \in \mathcal{S}^{\gamma}$ there exists a radial deformation $\widetilde{\mathbf{z}}^{\mathrm{rad}} \in \mathcal{S}^{\gamma}$ such that $E(\mathbf{u}) \geq E(\widetilde{\mathbf{z}}^{\mathrm{rad}})$, where

$$E(\mathbf{w}) = \int_A W(\nabla \mathbf{w}(\mathbf{x}))\, d\mathbf{x}.$$

Suppose further that $W(\mathbf{F}) \to \infty$ as $\det \mathbf{F} \to 0^+$, then there exists an absolute minimiser of E on $\mathcal{S}^{\gamma}$ that is radial.

Proof. First note that the above growth condition on $\widehat{W}$ implies the growth condition

$$\begin{aligned} W(\mathbf{F}) &\geq c\left(\frac{|\operatorname{adj}\mathbf{F}|^{\beta}}{(\det \mathbf{F})^{\beta-1}} + |\mathbf{F}|^{\gamma}\right) - k \det \mathbf{F} \\ &\geq c\frac{|\operatorname{adj}\mathbf{F}|^{\beta}}{(\det \mathbf{F})^{\beta-1}} + \bar{c}|\mathbf{F}|^{\gamma} - \bar{k} \end{aligned} \tag{8.4}$$

on the original energy W, where $\bar{c} > 0$ since $\gamma > 3$ (cf. (4.7)). It now follows from (8.4) and Proposition 7.2 that for any $\mathbf{u} \in \mathcal{S}^{\gamma}$ with finite energy $E(\mathbf{u})$, the corresponding inverse $\mathbf{v} = \mathbf{u}^{-1}$ lies in $W^{1,\beta}(A^*)$ and satisfies $\det \nabla \mathbf{v} > 0$ a.e. Next, apply Theorem 8.1 to $\widehat{W}$ and $\mathbf{v}$ to obtain the existence of a (modified) symmetrised map $\widetilde{\mathbf{v}}^{\mathrm{rad}}$ with no greater energy than

v. A final application of Proposition 7.2 shows that the inverse of $\widetilde{\mathbf{v}}^{\mathrm{rad}}$, which we denote $\widetilde{\mathbf{z}}^{\mathrm{rad}}$, satisfies $E(\mathbf{u}) \geq E(\widetilde{\mathbf{z}}^{\mathrm{rad}})$ as claimed.

Finally, the existence of a radial minimiser will follow if we apply the above argument to an absolute minimiser of E. It is clear from the assumptions we have made that W satisfies all of the required hypothesis of [1] (with the modifications of [6, pp. 371–377]) in order to deduce the existence of an absolute minimiser. (Note that polyconvexity of W is a consequence of the polyconvexity of $\widehat{W}$ as is proved in [2, pp. 210–211]. See, also [3, pp. 325–327].) □

Example 8.4. Let $\lambda > \mu$ and

$$W(\mathbf{F}) = \Psi\left(\frac{|\mathbf{F}|^{3/2}}{(\det\mathbf{F})^{1/2}}, \frac{|\operatorname{adj}\mathbf{F}|^3}{(\det\mathbf{F})^2}, \det\mathbf{F}\right),$$

where $\Psi : [0,\infty)\times[0,\infty)\times\mathbb{R}^+ \to \mathbb{R}$ is monotone increasing in its first two arguments and convex. Suppose further that Ψ satisfies the growth condition:

$$\Psi(s,t,j) \geq c\left[(s^2)^p + t^{\bar{p}} + \left(\frac{1}{j}\right)^{\bar{q}} + j^q\right] - k, \tag{8.5}$$

where c, p, q, $\bar{p}$, and $\bar{q}$ are strictly positive and satisfy $1/p+1/q<1$ and $1/\bar{p}+1/\bar{q}<1$. Then, for any $\mathbf{u}\in\mathcal{S}^p$, there exists a radial deformation $\widetilde{\mathbf{z}}^{\mathrm{rad}}\in\mathcal{S}^p$ that satisfies $E(\mathbf{u}) \geq E(\widetilde{\mathbf{z}}^{\mathrm{rad}})$.

Proof. This will follow from Theorem 8.3 once we verify that the hypotheses of the theorem are satisfied. First note that the growth assumptions (8.5), together with Young's inequality, guarantee that W satisfies a growth condition of the form

$$W(\mathbf{F}) \geq c^*\left(|\mathbf{F}|^{\hat{p}} + \frac{|\operatorname{adj}\mathbf{F}|^{\hat{q}}}{(\det\mathbf{F})^{\hat{q}-1}}\right) - k^*, \tag{8.6}$$

for some constants $\hat{p}>3$, $\hat{q}>3$, $c^*>0$, and k^*. Next, the stored-energy function

$$\widehat{W}(\mathbf{F}) = (\det\mathbf{F})\,W(\mathbf{F}^{-1}) = (\det\mathbf{F})\,G\left(\frac{|\operatorname{adj}\mathbf{F}|^{3/2}}{\det\mathbf{F}}, \frac{|\mathbf{F}|^3}{\det\mathbf{F}}, \frac{1}{\det\mathbf{F}}\right)$$

is jointly convex in the variables $(|\operatorname{adj}\mathbf{F}|^{3/2}, |\mathbf{F}|^3, \det\mathbf{F})$ (see Example 6.4 and Footnote 7) and, by (8.6), satisfies the growth estimate

$$\begin{aligned}\widehat{W}(\mathbf{F}) = (\det\mathbf{F})W(\mathbf{F}^{-1}) &\geq c^*\left(\frac{|\operatorname{adj}\mathbf{F}|^{\hat{p}}}{(\det\mathbf{F})^{\hat{p}-1}} + |\mathbf{F}|^{\hat{q}}\right) - k^*\det\mathbf{F}\\ &\geq c^*\frac{|\operatorname{adj}\mathbf{F}|^{\hat{p}}}{(\det\mathbf{F})^{\hat{p}-1}} + \bar{c}^*|\mathbf{F}|^{\hat{q}} - \bar{k}^*,\end{aligned}$$

where $\bar{c}^*>0$. Thus we may apply Theorem 8.3 to this class of stored-energy functions to obtain the claimed result. □

A Appendix

Lemma A.1. *Let $\phi \in C^2(\mathbb{R}^+;\mathbb{R})$ be convex. Then*

$$Q(s,t) = t\phi\left(\frac{s}{t}\right)$$

is convex on $\mathbb{R}^+ \times \mathbb{R}^+$.

Proof. The partial derivatives of Q are

$$Q_s = \phi', \qquad Q_t = \phi - \frac{s}{t}\phi',$$

$$Q_{ss} = \frac{1}{t}\phi'', \qquad Q_{st} = -\frac{s}{t^2}\phi'', \qquad Q_{tt} = \frac{s^2}{t^3}\phi''.$$

Since ϕ is convex both Q_{ss} and Q_{tt} are non-negative. The convexity of Q then follows from the identity $Q_{ss}Q_{tt} = Q_{st}^2$, which is clear. □

Lemma 5.1. *Let $f : [a,b] \to \mathbb{R}$ be (absolutely) continuous. Define $g : [a,b] \to \mathbb{R}$ by*

$$g(x) := \inf_{t\in[a,x]} f(t). \tag{A.1}$$

Then g is (absolutely) continuous and monotone decreasing. Moreover, there exist a countable sequence of pairwise-disjoint open intervals, $(a_k, b_k) \subset (a,b)$ such that $g \equiv g_k$ is constant on each interval (a_k, b_k), where $g_k \leq f(a)$, and $g = f$ on $K := (a,b) \setminus \cup_k (a_k, b_k)$. Additionally, if f and g are absolutely continuous then

$$g' = f'\chi_K \quad a.e.\ on \quad [a,b] \tag{A.2}$$

and if f is Lipschitz continuous then so is g.

Proof. It is clear from (A.1) that g is monotone decreasing with $g(z) \leq f(a)$ for all $z \in [a,b]$. Assume now that f is continuous and note that by definition of g

$$f(z) \geq g(z) \quad \text{for all} \quad z \in [a,b]. \tag{A.3}$$

Next, fix $x \in (a,b]$ and let $x_n \in [a,b]$ be a strictly monotone increasing sequence of points that converges to x. Then for any n

$$\begin{aligned} g(x) = \inf_{t\in[a,x]} f(t) &= \min\{\inf_{t\in[a,x_n]} f(t), \inf_{t\in[x_n,x]} f(t)\} \\ &= \min\{g(x_n), \inf_{t\in[x_n,x]} f(t)\}. \end{aligned}$$

If there exists an n such that $g(x) = g(x_n)$ then the monotonicity of g implies that it is constant on $[x_n, x]$. Thus g will be continuous from the left at x. Otherwise, since f is continuous,

$$g(x) = \inf_{t\in[x_n,x]} f(t) \to f(x) \text{ as } x_n \nearrow x,$$

and hence $g(x) = f(x)$. However, in view of (A.3) and the monotonicity of g

$$f(x_n) \geq g(x_n) \geq g(x) = f(x)$$

and so the continuity of f yields the continuity of g from the left. The proof from the right at any $x \in [a, b)$ is similar.

To construct the required open intervals define

$$U := \{x \in (a, b) : \exists \varepsilon_x > 0 \text{ such that } g(x + s) = g(x) \text{ for all } s \in (-\varepsilon_x, \varepsilon_x)\}.$$

Then since g is continuous, it follows that U is an open subset of (a, b). A standard result on the topology of the real line then yields a countable sequence of pairwise-disjoint intervals (a_k, b_k) that satisfy

$$U = \bigcup_k (a_k, b_k).$$

Moreover, since g is constant in a neighbourhood of each point in U it is clear that $g \equiv g_k$, a constant, on each open interval (a_k, b_k). In addition, since $g \leq f(a)$ on $[a, b]$ it follows that $g_k \leq f(a)$.

To see that $g = f$ on $K := (a, b)\backslash U$ let $x \in K$ and suppose once again that $x_n \in [a, x]$ is a strictly monotone increasing sequence of points that converges to x. Then by the previous argument either g is constant on some interval $[x_n, x]$ or $f(x) = g(x)$, as required. A similar argument with a strictly monotone decreasing sequence of points $x^k \searrow x$ shows that either g is constant on some interval $[x, x^k]$ or $f(x) = g(x)$, as required. Finally, if g were constant on $[x_n, x^k]$ with $x_n < x < x^k$ then $x \in U$, which is not possible.

Now assume in addition that f is absolutely continuous. We note that g is monotone and hence has bounded variation. Since g is also continuous a sufficient (and necessary) condition for g to be absolutely continuous is that it satisfies Lusin's N-condition.[10] This property is clear for g since

$$g(A) = g(A \cap U) \cup g(A \cap K) = g(A \cap U) \cup f(A \cap K).$$

Thus if A has measure zero then $f(A \cap K)$ has measure zero by the absolutely continuity of f and $g(A \cap U)$ has measure zero since it consists of (at most) a countable number of real numbers g_k.

[10]That is, g maps Lebesgue *null* sets into such sets. See, e.g., [15, Chapter IX].

To prove (A.2) we first note that $g(x) \equiv g_k$ for $x \in (a_k, b_k)$ and hence $g' \equiv 0$ on U, which establishes (A.2) for $x \in U$. Next, the absolute continuity of f and g yield their differentiability almost everywhere. Now let $x \in K$ be a point at which both f and g are differentiable and note that $f(x) = g(x)$. If x is a limit point of K then there is a sequence $x_i \to x$ with $x_i \in K$ and $x_i \neq x$. Thus $f(x_i) = g(x_i)$ and hence

$$f'(x) = \lim_{i\to\infty} \frac{f(x_i) - f(x)}{x_i - x} = \lim_{i\to\infty} \frac{g(x_i) - g(x)}{x_i - x} = g'(x),$$

since both functions are differentiable at x. This completes the proof of (A.2) since the isolated points of a bounded set are countable.

Finally, if f is Lipschitz continuous with Lipschitz constant M then $|f'| \le M$ and hence, in view of (A.2), $|g'| \le M$ and so g is Lipschitz continuous with Lipschitz constant M. This completes the proof of the lemma. □

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References

[1] J. M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rational Mech. Anal.* **63** (1977), 337–403.

[2] J. M. Ball, *Constitutive inequalities and existence theorems in nonlinear elastostatics* in Nonlinear Analysis and Mechanics: Heriot-Watt Symposium, Vol. 1, R. J. Knops ed., Pitman, 1977, 187–241,

[3] J. M. Ball, Global invertibility of Sobolev functions and the interpenetration of matter, *Proc. Roy. Soc. Edinburgh* **88A** (1981), 315–328.

[4] J. M. Ball, Discontinuous equilibrium solutions and cavitation in nonlinear elasticity, *Phil. Trans. R. Soc. Lond.* A **306** (1982), 557–611.

[5] P. G. Ciarlet and J. Nečas, Injectivity and self-contact in non-linear elasticity, *Arch. Rational Mech. Anal.*, **97** (1987), 171–188.

[6] P. G. Ciarlet, *Mathematical Elasticity, vol. 1*, Elsevier, 1988.

[7] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, 1992.

[8] H. Federer, *Geometric Measure Theory*, Springer, 1969.

[9] I. Fonseca and W. Gangbo, *Degree Theory in Analysis and Applications*, Clarendon Press, 1995.

[10] R. D. James and S. J. Spector, The formation of filamentary voids in solids, *J. Mech. Phys. Solids* **39** (1991), 783–813.

[11] M. Marcus and V. J. Mizel, Transformations by functions in Sobolev spaces and lower semicontinuity for parametric variational problems, *Bull. Am. Math. Soc.* **79** (1973), 790–795.

[12] S. Müller and S. J. Spector, An existence theory for nonlinear elasticity that allows for cavitation, *Arch. Rational Mech. Anal.* **131** (1995), 1–66.

[13] S. Müller, S. J. Spector, and Q. Tang, Invertibility and a topological property of Sobolev maps, *SIAM J. Math. Anal.* **27** (1996), 959–976.

[14] S. Müller, Q. Tang, and B. S. Yan, On a new class of elastic deformation not allowing for cavitation, *Analyse non linéaire* **11** (1994), 217–243.

[15] I. P. Natanson, *Theory of Functions of a Real Variable, Vol. I*, Frederick Unger, New York, 1955.

[16] J. T. Schwartz, *Nonlinear Functional Analysis*, Gordon and Breach, 1969.

[17] J. Sivaloganathan, Uniqueness of regular and singular equilibria for spherically symmetric problems of nonlinear elasticity, *Arch. Rational Mech. Anal.* **96** (1986), 97–136.

[18] J. Sivaloganathan and S. J. Spector, On the Symmetry of Energy Minimising Deformations in Nonlinear Elasticity I: Incompressible Materials.

[19] V. Šverák, Regularity properties of deformations with finite energy, *Arch. Rational Mech. Anal.* **100** (1988), 105–127.

[20] Q. Tang, Almost-everywhere injectivity in nonlinear elasticity, *Proc. Roy. Soc. Edinburgh* **109A** (1988), 79–95.

[21] W. P. Ziemer, *Weakly Differentiable Functions*, Springer, 1989.