On the Uniqueness of Energy Minimizers in Finite Elasticity

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Abstract The uniqueness of absolute minimizers of the energy of a compressible, hyperelastic body subject to a variety of dead-load boundary conditions in two and three dimensions is herein considered. Hypotheses under which a given solution of the corresponding equilibrium equations is the unique absolute minimizer of the energy are obtained. The hypotheses involve uniform polyconvexity and pointwise bounds on derivatives of the stored-energy density when evaluated on the given equilibrium solution. In particular, an elementary proof of the uniqueness result of Fritz John (Comm. Pure Appl. Math. **25**, 617–634, 1972) is obtained for uniformly polyconvex stored-energy densities.

Keywords Finite Elasticity · Nonlinear Elasticity · Uniqueness · Equilibrium Solutions · Energy Minimizers · Nonuniqueness · Uniform Polyconvexity · Strict Polyconvexity · Strongly Polyconvex

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1 Introduction

In this manuscript we consider the uniqueness of absolute minimizers of the energy of a compressible, hyperelastic body under dead loads. Although one does not always expect such uniqueness, for example, when a thin rod is subjected to uniaxial compression there should be more than one buckled minimizer, a result of Zhang [63] for the displacement problem shows that there is exactly one absolute minimizer of the elastic energy for certain boundary displacements.

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In addition to the displacement problem we also consider both the traction and the mixed problem for energy functions that are uniformly polyconvex, that is, when $n = 3$, stored-energy densities of the form

$$W(\mathbf{x},\mathbf{F}) = \frac{\omega(\mathbf{x})}{p}|\mathbf{F}|^p + \Phi(\mathbf{x},\mathbf{F},\operatorname{cof}\mathbf{F},\det\mathbf{F}),$$

where $\omega(\mathbf{x}) \geq \omega_o > 0$, $p \geq 3$, $\mathbf{N} \mapsto \Phi(\mathbf{x},\mathbf{N})$ is convex,[1] $\det\mathbf{F}$ denotes the determinant of the 3 by 3 matrix $\mathbf{F}$, $\operatorname{cof}\mathbf{F}$ its cofactor matrix, and $|\mathbf{F}|$ the square-root of the sum of the squares of the elements of $\mathbf{F}$. Our main result, Theorem 4.2, shows (using elementary methods) that for such energies any (weak) solution of the equilibrium equations that satisfies a certain pointwise bound will be the unique absolute minimizer of the energy. Moreover, there can be no other solution of the equilibrium equations that satisfies this bound.

We mention that our results do not require the material to be either homogeneous or isotropic. Our main theorem applies just as well to a heterogeneous material such as a composite. (See Remark 4.3 and the examples in §6.3.) We note, in Remark 4.5, that our proof of Theorem 4.2 is also valid when Φ is not globally convex. If instead a weak solution of the equilibrium equations lies at a point of convexity of Φ (see (4.16)) and satisfies the required pointwise bound, then that deformation must be a, potentially nonunique, absolute minimizer of the energy. Theorem 4.2 therefore has implications for stored-energies that admit phase transitions (see Ball and James [6] or, e.g., Grabovsky and Truskinovsky [27] and the references therein).

In the special case when the stored-energy density W of a homogeneous body $\mathscr{B} \subset \mathbb{R}^n$, $n = 2$ or $n = 3$, is given by

$$W(\mathbf{F}) = \frac{\omega_o}{n}|\mathbf{F}|^n + \Phi(\mathbf{F},\det\mathbf{F}), \tag{1.1}$$

where $\omega_o > 0$ and Φ is convex, then our results show that if $\mathbf{u}_e$ is a weak solution of the equilibrium equations that satisfies (see (4.6)–(4.8) and Remark A.2)

$$||\Lambda(\nabla\mathbf{u}_e,\det\nabla\mathbf{u}_e)||_{L^\infty(\mathscr{B})} < \omega_o/(2n-3), \qquad \Lambda(\mathbf{F},\det\mathbf{F}) := \frac{\partial}{\partial\lambda}\Phi(\mathbf{F},\lambda)\Big|_{\lambda=\det\mathbf{F}}, \tag{1.2}$$

then $\mathbf{u}_e$ is the unique absolute minimizer of the elastic energy. Moreover, no other solution of the equilibrium equations can satisfy $(1.2)_1$. Furthermore, for the pure-displacement problem, Theorem 4.2 also shows that the same results are valid if $(1.2)_1$ is replaced by

$$||\Lambda(\nabla\mathbf{u}_e,\det\nabla\mathbf{u}_e) - \nu||_{L^\infty(\mathscr{B})} < \omega_o/(2n-3), \tag{1.3}$$

where $\nu \in \mathbb{R}$ denotes an arbitrary constant.

For the displacement problem we obtain additional results. We first show, in Theorem 5.1, that a result of Zhang [63] is a simple consequence of our main theorem; we prove that if an equilibrium solution, $\mathbf{u}_e$, is sufficiently close to a homogeneous deformation (in the Sobolev space $W^{1,\infty}$), then $\mathbf{u}_e$ is the unique absolute minimizer of the energy and there are no other equilibrium solutions nearby.

We also consider, in Theorem 5.3, a uniqueness result of John [34], who proved that there is at most one equilibrium solution with (sufficiently) small strain: $\mathbf{E} := (\nabla\mathbf{u})^{\mathrm{T}}\nabla\mathbf{u} - \mathbf{I}$. We use a recent result of Šilhavý [49], which produces a polyconvex representative that is invariant under rotations, to show that John's result is a direct consequence of our proof of

[1] Here $\mathbf{N} = (\mathbf{A},\mathbf{B},t)$ for some 3 by 3 matrices $\mathbf{A}$ and $\mathbf{B}$ and some scalar $t > 0$.

the above-mentioned result of Zhang. We thus provide an elementary proof of a version[2] of the result in [34] that does not require the use and properties of BMO [35].

In §6 we present some illustrative examples of equilibrium solutions that satisfy the hypotheses of our theorems. We consider the extension of a 2-dimensional, rectangular body and show that, for both a mixed problem and a pure-traction problem, the homogeneous solution is the unique absolute minimizer of the energy when the stored-energy density is compressible neo-Hookean (see (6.2)). We also consider an annulus composed of two distinct compressible neo-Hookean materials and show that, for small radial boundary displacements, the resulting radial minimizer is the unique absolute minimizer of the energy. We note that a similar analysis will work for an annulus composed of many different layers that are not necessarily compressible neo-Hookean. In §7 we briefly mention an alternative approach to the uniqueness of minimizers due to Gao, Neff, Roventa, and Thiel [24].

We note that it is not clear from our proofs whether or not a condition such as $(1.2)_1$ or (1.3) is necessary for the uniqueness of absolute minimizers. However, we do observe that our results together with the construction of two (or more) equilibrium solutions with equal energy, such as occurs in the buckling of a rectangular rod (see, e.g., [50]) or the twisting of an annulus (see, e.g., [44]), implies that neither solution can satisfy $(1.2)_1$ or (1.3). See Remarks 4.4, 6.1, and 6.4.

Most of the prior literature on uniqueness in finite elasticity considers the uniqueness of equilibrium solutions rather than energy minimizers. For example, results of Gurtin and Spector [31] imply that there is at most one solution of the equilibrium equations that lies in any convex set where the second variation of the energy is strictly positive. Knops and Stuart [37] (also see Bevan [9] and Taheri [59]) have shown that, for a star-shaped body, a homogeneous deformation is the unique smooth equilibrium solution that satisfies a homogeneous pure-displacement boundary condition whenever the energy is strictly quasiconvex at that deformation and globally rank-one convex.

Alternatively, there are a number of results that establish the nonuniqueness of equilibrium solutions for compressible materials.[3] For example, Post and Sivaloganathan [44] (verifying a conjecture of John [33,34]) proved that there are (at least) a countably infinite number of equilibrium solutions for certain pure-displacement problems for an annulus. Antman [1] has shown that, for the pure-traction problem, a thick spherical shell without loads has a second equilibrium solution corresponding to an everted deformation. Simpson and Spector [50] have proven that, in addition to the homogeneous equilibrium solution, there are indeed two distinct buckled equilibrium solutions for certain 2-dimensional isotropic bars subject to uniaxial compression.

We note that there is an interesting result of Spadaro [54] for the pure-displacement problem in 2-dimensions for constitutive relations of the form (1.1) with $n = 2$. Spadaro showed that there must be at least two absolute minimizers of the energy to a certain boundary-value problem when the body is a disk. However (as he notes), his construction is not compatible with finite elasticity, since it requires negative Jacobians.

Finally, we mention some interesting open problems suggested by our analysis. Firstly, we have not considered incompressible elastic bodies, however, we suspect that results similar to ours should hold. A good place to start would be the results of Zhang [63], which were established for such bodies. Secondly, we have only considered dead loads. A uniqueness result for live loads would be of interest. Here one might want to look at [13,43,46,

[2] Our hypotheses on the stored-energy differs from that in [34]. See Remark 5.4.

[3] For interesting examples of nonuniqueness for both compressible and incompressible materials see, e.g., [3, §9], [14, §5.8], [1,8,29–31,44,45], and the references therein.

55], [14, §2.7], or [48, §13.3]. Lastly, it would be of interest to extend John's [34] uniqueness result (as well as Zhang's [63]) for the displacement problem to the mixed and traction problems. Our initial thought was that the Geometric Rigidity Theory of Friesecke, James, and Müller [23] (see, also, Kohn [36] and Conti and Schweizer [17]) might be useful for such an analysis, however, one of the referees of this paper suggested that a technique similar to that of Kristensen and Taheri [38] as well as the paper of Fefferman and Stein [22] could also prove helpful.

2 Preliminaries; The Nonlinear Problem

2.1 Preliminaries

We consider a body that for convenience we identify with the region $\overline{\mathscr{B}} \subset \mathbb{R}^n$, $n = 2$ or $n = 3$, that it occupies in a fixed reference configuration. We assume that $\mathscr{B} \neq \varnothing$ is a connected, bounded, open set whose boundary, $\partial\mathscr{B}$, is Lipschitz[4] (see, e.g., [21]). A *deformation* of $\mathscr{B}$ is a mapping that lies in the space

$$\mathrm{Def} := \{\mathbf{u} \in W^{1,1}(\mathscr{B};\mathbb{R}^n) : \det\nabla\mathbf{u} > 0 \ a.e.\},$$

where $\det\mathbf{F}$ denotes the determinant of $\mathbf{F} \in \mathrm{Lin}_n$ (the space of linear maps from $\mathbb{R}^n$ into $\mathbb{R}^n$) and for $1 \le p \le \infty$, $W^{1,p}(\mathscr{B};\mathbb{R}^n)$ denotes the usual Sobolev space of (Lebesgue) measurable (vector-valued) functions $\mathbf{u} \in L^p(\mathscr{B};\mathbb{R}^n)$ whose distributional derivative, $\nabla\mathbf{u}$, is also contained in L^p. We write δ_{ij} for the Kronecker delta: thus,

$$\delta_{ij} := \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

We assume that the body is composed of a hyperelastic material whose *stored-energy density* $W : \overline{\mathscr{B}} \times \mathrm{Lin}_n \to [0,\infty]$ with $\mathbf{x} \mapsto W(\mathbf{x},\mathbf{F})$ (Lebesgue) measurable[5] for every $\mathbf{F} \in \mathrm{Lin}_n$. $W(\mathbf{x},\nabla\mathbf{u}(\mathbf{x}))$ gives the elastic energy stored at almost every point $\mathbf{x} \in \mathscr{B}$ of a deformation $\mathbf{u} \in \mathrm{Def}$. We assume that the response of the material is *invariant under a change in observer* and hence that, for $a.e.$ $\mathbf{x} \in \mathscr{B}$,

$$W(\mathbf{x},\mathbf{QF}) = W(\mathbf{x},\mathbf{F}) \quad \text{for every } \mathbf{F} \in \mathrm{Lin}_n^{\succ} \text{ and } \mathbf{Q} \in \mathrm{Orth}_n^{\succ}, \tag{2.1}$$

where $\mathrm{Lin}_n^{\succ}$ denotes those $\mathbf{F} \in \mathrm{Lin}_n$ with $\det\mathbf{F} > 0$ and $\mathrm{Orth}_n^{\succ}$ denotes those $\mathbf{Q} \in \mathrm{Lin}_n^{\succ}$ that satisfy $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}$ ($\mathbf{I} \in \mathrm{Lin}_n^{\succ}$ denotes the *identity*, i.e, $\mathbf{Ia} = \mathbf{a}$ for all $\mathbf{a} \in \mathbb{R}^n$).

We further assume that, for $a.e.$ $\mathbf{x} \in \mathscr{B}$,

$$\mathbf{F} \mapsto W(\mathbf{x},\mathbf{F}) \in C(\mathrm{Lin}_n;[0,\infty]) \cap C^1(\mathrm{Lin}_n^{\succ};\mathbb{R}^{\succeq}),$$
$$\lim_{|\mathbf{F}|\to\infty} W(\mathbf{x},\mathbf{F}) = \lim_{\det\mathbf{F}\to 0^+} W(\mathbf{x},\mathbf{F}) = +\infty,$$
$$W(\mathbf{x},\mathbf{F}) = +\infty \quad \text{if and only if} \quad \det\mathbf{F} \le 0,$$

where $\mathbb{R}^{\succeq} := [0,\infty)$. The (Piola-Kirchhoff) *stress* is then the derivative

$$\mathbf{S}(\mathbf{x},\mathbf{F}) := \frac{\partial}{\partial\mathbf{F}} W(\mathbf{x},\mathbf{F}) : \overline{\mathscr{B}} \times \mathrm{Lin}_n^{\succ} \to \mathrm{Lin}_n,$$

for $a.e.$ $\mathbf{x} \in \mathscr{B}$. We call the body *homogeneous* if the stored-energy function W is independent of $\mathbf{x}$. We call the reference configuration *stress free* if, for $a.e.$ $\mathbf{x} \in \mathscr{B}$, $\mathbf{S}(\mathbf{x},\mathbf{I}) = \mathbf{0}$.

[4] This assumption allows for a piecewise C^1 boundary, for example, a rectangle.

[5] In particular, the stored-energy density may therefore be piecewise continuous.

2.2 The Nonlinear Problem

We assume the body is subject to dead loads. We take

$$\partial\mathscr{B} = \overline{\mathscr{D}} \cup \overline{\mathscr{S}} \quad \text{with } \mathscr{D} \text{ and } \mathscr{S} \text{ relatively open and } \mathscr{D} \cap \mathscr{S} = \varnothing.$$

If $\mathscr{D} \neq \varnothing$ we assume that a function $\mathbf{d} \in C(\overline{\mathscr{D}};\mathbb{R}^n)$ is prescribed; $\mathbf{d}$ will give the deformation of $\mathscr{D}$. If $\mathscr{S} \neq \varnothing$ we assume that a function $\mathbf{s} \in L^1(\mathscr{S};\mathbb{R}^n)$ is prescribed; for $\mathscr{H}^{n-1}$-*a.e.* $\mathbf{x} \in \mathscr{S}$, $\mathbf{s}(\mathbf{x})$ will give the surface force (per unit area, when $n=3$, and per unit length, when $n=2$) exerted on the body, at the point $\mathbf{x}$, by its environment. Finally, we suppose that a function $\mathbf{b} \in L^1(\mathscr{B};\mathbb{R}^n)$ is prescribed; for *a.e.* $\mathbf{x} \in \mathscr{B}$, $\mathbf{b}(\mathbf{x})$ will give the body force (per unit volume, when $n=3$, and per unit area, when $n=2$) exerted on the body, at the point $\mathbf{x}$, by its environment. Here, and in the sequel, $\mathscr{H}^k$ denotes k-dimensional Hausdorff measure. The set of *admissible deformations* will be denoted by

$$\mathscr{A} := \{\mathbf{u} \in \mathrm{Def} \cap W^{1,n}(\mathscr{B};\mathbb{R}^n) \cap C(\overline{\mathscr{B}};\mathbb{R}^n) : \mathbf{u} = \mathbf{d} \text{ on } \mathscr{D}\}.$$

Remark 2.1 A result of Vodop′yanov and Gol′dšhteĭn [61] (see, also, [58, Theorem 4]) implies that each $\mathbf{u} \in W^{1,n}(\mathscr{B};\mathbb{R}^n)$ with strictly positive Jacobian has a continuous representative. Thus, discontinuities such as cavitation (see, e.g., Ball [4]) are not allowed in this manuscript.

The *total energy* E of a deformation $\mathbf{u} \in \mathscr{A}$ is defined by

$$E(\mathbf{u}) := \int_{\mathscr{B}} \Big[W\big(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x})\big) - \mathbf{b}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \Big] \mathrm{d}\mathbf{x} - \int_{\mathscr{S}} \mathbf{s}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \, \mathrm{d}\mathscr{H}^{n-1}_{\mathbf{x}}. \tag{2.2}$$

Under suitable additional hypotheses on W one might hope to show that any $\mathbf{u}$ that is a minimizer (local in an appropriate topology or global) of E has first variation zero, i.e.,

$$0 = \int_{\mathscr{B}} \Big[\mathbf{S}\big(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x})\big) : \nabla\mathbf{w}(\mathbf{x}) - \mathbf{b}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) \Big] \mathrm{d}\mathbf{x} - \int_{\mathscr{S}} \mathbf{s}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) \, \mathrm{d}\mathscr{H}^{n-1}_{\mathbf{x}} \tag{2.3}$$

for all *variations* $\mathbf{w} \in \mathrm{Var}$, where

$$\mathrm{Var} := \{\mathbf{w} \in W^{1,n}(\mathscr{B};\mathbb{R}^n) \cap C(\overline{\mathscr{B}};\mathbb{R}^n) : \mathbf{w} = \mathbf{0} \text{ on } \mathscr{D}\},$$

$\mathbf{F} : \mathbf{G} := \mathrm{tr}(\mathbf{F}\mathbf{G}^{\mathrm{T}})$, $\mathrm{tr}\,\mathbf{M}$ denotes the trace of $\mathbf{M} \in \mathrm{Lin}_n$, and $\mathbf{M}^{\mathrm{T}}$ denotes its transpose.

Moreover, one would then want to show that $\mathbf{u}$ is a *classical solution of the equations of equilibrium*, that is,[6] $\mathbf{u} \in C^2(\mathscr{B};\mathbb{R}^n) \cap C^1(\overline{\mathscr{B}};\mathbb{R}^n) \cap \mathscr{A}$ satisfies

$$\mathrm{Div}\,\mathbf{S}\big(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x})\big) + \mathbf{b}(\mathbf{x}) = \mathbf{0} \quad \text{for } \mathbf{x} \in \mathscr{B} \tag{2.4}$$

and the *traction boundary conditions*

$$\mathbf{S}\big(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x})\big)\mathbf{n}(\mathbf{x}) = \mathbf{s}(\mathbf{x}) \quad \text{for } \mathbf{x} \in \mathscr{S}. \tag{2.5}$$

Unfortunately, such results[7] have not been obtained for arbitrary minimizers. In general, *in this manuscript we will therefore assume that one or more solutions of* (2.3) *are given.*

[6] If $\mathscr{S} = \varnothing$, then $\mathbf{u} \in C^2(\mathscr{B};\mathbb{R}^n) \cap \mathscr{A}$ suffices.

[7] In general, one can only prove that a minimizer is a weak solution of alternative forms of the equilibrium equations. See [5, Theorem 2.4] and the references therein. However, Lemma 2.9 shows that additional hypotheses may imply that a minimizer is in fact a weak equilibrium solution.

Remark 2.2 There are a number of well-known classical equilibrium solutions that are of interest. Among these are:

1. Homogeneous solutions (see Remarks 2.7 and 4.8, Proposition 4.7, and §6.2);
2. Solutions obtained using the implicit function theorem (see Remark 5.2); and
3. Radial solutions when the body, in its reference configuration, is a disk, an annulus, a ball, or a thick spherical shell (see §6.1 and §6.3).

Definition 2.3 We say that $\mathbf{u}_e \in \mathscr{A}$ is a *weak equilibrium solution* if $E(\mathbf{u}_e) < +\infty$,

$$\mathbf{x} \mapsto \mathbf{S}\big(\mathbf{x}, \nabla\mathbf{u}_e(\mathbf{x})\big) \in L^{n'}(\mathscr{B};\mathrm{Lin}_n), \qquad n' := \frac{n}{n-1}, \tag{2.6}$$

$\mathbf{u}_e$ satisfies (2.3) for all $\mathbf{w} \in \mathrm{Var}$, and if $\mathscr{D} = \varnothing$,

$$\int_{\mathscr{B}} \big[\mathbf{u}_e(\mathbf{x}) - \mathbf{x}\big]\mathrm{d}\mathbf{x} = \mathbf{0}. \tag{2.7}$$

If $\mathscr{D} = \partial\mathscr{B}$ we will call $\mathbf{u}_e$ a weak solution of the *(pure) displacement problem*. If $\mathscr{S} = \partial\mathscr{B}$ we will call $\mathbf{u}_e$ a weak solution of the *(pure) traction problem*. Otherwise, we will refer to such a $\mathbf{u}_e$ as a weak solution of the *(genuine) mixed problem*.

Remark 2.4 When $\mathscr{S} = \partial\mathscr{B}$ any translation of a weak equilibrium solution $\mathbf{u}_e$ will satisfy both (2.3) and (2.6). Equation (2.7) eliminates this nonuniqueness.

Remark 2.5 Our assumption that $\mathbf{S} \in L^{n'}$ is, in general, more stringent than expected for an absolute minimizer $\mathbf{u} \in W^{1,n}$ of E. However, it is necessitated by (2.3) which requires $\mathbf{S} : \nabla\mathbf{w}$ to be integrable for $\mathbf{w} \in \mathrm{Var} \subset W^{1,n}$. As will become evident, our conditions for uniqueness, e.g., (4.6) and (4.7), may sometimes require a weak equilibrium solution $\mathbf{u}_e$ to satisfy $\mathbf{u}_e \in W^{1,\infty}(\mathscr{B};\mathbb{R}^n)$. See Remark 4.6.

Remark 2.6 Any classical solution of (2.4) and (2.5) is also a weak equilibrium solution.

Remark 2.7 Let the body be homogeneous and $\mathscr{D} \neq \varnothing$. Fix $\mathbf{F}_e \in \mathrm{Lin}_n^{\succ}$, $\mathbf{a} \in \mathbb{R}^n$, and define $\mathbf{u}_e(\mathbf{x}) := \mathbf{F}_e\mathbf{x} + \mathbf{a}$ for all $\mathbf{x} \in \overline{\mathscr{B}}$. Then $\mathbf{u}_e(\mathbf{x}) = \mathbf{d}(\mathbf{x}) := \mathbf{F}_e\mathbf{x} + \mathbf{a}$ for all $\mathbf{x} \in \mathscr{D}$. If $\mathscr{S} \neq \varnothing$ assume, in addition, that $\mathbf{s}(\mathbf{x}) := \mathbf{S}(\mathbf{F}_e)\mathbf{n}(\mathbf{x})$ for $\mathscr{H}^{n-1}$-*a.e.* $\mathbf{x} \in \mathscr{S}$, where, for such $\mathbf{x}$, $\mathbf{n}(\mathbf{x})$ denotes the outward unit normal to the boundary. Then $\mathbf{u}_e$ is an admissible deformation that satisfies both the equilibrium equations (2.4) (with $\mathbf{b} \equiv \mathbf{0}$) and the traction boundary conditions (2.5); thus $\mathbf{u}_e$ is a classical equilibrium solution.

Although it is not known if an arbitrary minimizer of the energy is a solution of the equilibrium equations, it will be if the mapping happens to satisfy certain additional conditions. In order to illustrate this we first formally define what we mean by a local minimizer.

Definition 2.8 Let $\mathbf{u}_m \in \mathscr{A}$. We say that $\mathbf{u}_m$ is a *weak relative minimizer*[8] of the energy E provided that there exists a $\delta > 0$ such that

$$E(\mathbf{u}_m) \leq E(\mathbf{u}_m + \mathbf{w})$$

for all variations $\mathbf{w} \in \mathrm{Var} \cap W^{1,\infty}(\mathscr{B};\mathbb{R}^n)$ that satisfy $||\mathbf{w}||_{L^\infty(\mathscr{B})} + ||\nabla\mathbf{w}||_{L^\infty(\mathscr{B})} < \delta$.

[8] See, e.g., Del Piero and Rizzoni [19] and the references therein for results concerning weak relative minimizers in Elasticity. See Grabovsky and Mengesha [25,26] for results concerning the relationship between such minimizers and strong relative minimizers, although not for Elasticity.

The next lemma then illustrates a circumstance where such a $\mathbf{u}_{\mathrm{m}}$ does indeed satisfy the equilibrium equations. The proof follows from the mean-value theorem together with the bounded convergence theorem (see, e.g., Ball [5, §2.4] or [18, §3.4.2]).

Lemma 2.9 *Let $\mathbf{u}_{\mathrm{m}} \in \mathscr{A} \cap W^{1,\infty}(\mathscr{B};\mathbb{R}^n)$ be a weak relative minimizer of E that satisfies*

$$\det \nabla \mathbf{u} > \varepsilon \quad a.e. \tag{2.8}$$

for some $\varepsilon > 0$. Suppose, in addition, that either

$$\mathbf{S} \in C(\overline{\mathscr{B}} \times \mathrm{Lin}_n^{\succ}) \ \textit{ or } \ \mathbf{S} \textit{ is bounded on compact subsets of } \overline{\mathscr{B}} \times \mathrm{Lin}_n^{\succ}. \tag{2.9}$$

Then $\mathbf{u}_{\mathrm{m}}$ is a weak equilibrium solution.

3 Uniform Polyconvexity

Let $n = 2$ or $n = 3$. Define

$$\mathscr{E}_2 := \mathrm{Lin}_2, \qquad \mathscr{E}_3 := \mathrm{Lin}_3 \times \mathrm{Lin}_3,$$
$$\mathscr{E}_2^{\succ} := \mathrm{Lin}_2^{\succ}, \quad \mathscr{E}_3^{\succ} := \mathrm{Lin}_3^{\succ} \times \mathrm{Lin}_3^{\succ}.$$

We assume that the stored-energy density is *uniformly polyconvex,*[9] that is, there is a constant $p \geq n$ and functions $\omega : \mathscr{B} \to \mathbb{R}^{\succ}$ and $\Phi^{(n)} : \mathscr{B} \times \mathscr{E}_n \times \mathbb{R}^{\succ} \to \mathbb{R}$, $\mathbb{R}^{\succ} := (0,\infty)$, that satisfy, for all $\mathbf{F} \in \mathrm{Lin}^{\succ}$ and $a.e.$ $\mathbf{x} \in \mathscr{B}$,

$$W(\mathbf{x},\mathbf{F}) = \frac{\omega(\mathbf{x})}{p}|\mathbf{F}|^p + \begin{cases} \Phi^{(2)}(\mathbf{x},\mathbf{F},\det\mathbf{F}), & \text{if } n = 2, \\ \Phi^{(3)}(\mathbf{x},\mathbf{F},\mathrm{cof}\,\mathbf{F},\det\mathbf{F}), & \text{if } n = 3, \end{cases} \tag{3.1}$$

where $|\mathbf{F}| := \sqrt{\mathbf{F} : \mathbf{F}}$,

I. $\omega \in L^\infty(\mathscr{B})$ satisfies $\omega \geq \omega_{\mathrm{o}}$ for some constant $\omega_{\mathrm{o}} > 0$;
II. $\mathbf{x} \mapsto \Phi^{(n)}(\mathbf{x},\mathbf{M},\lambda)$ is measurable for every $\mathbf{M} \in \mathscr{E}_n$ and $\lambda > 0$; and, for $a.e.$ $\mathbf{x} \in \mathscr{B}$,
III. $(\mathbf{M},\lambda) \mapsto \Phi^{(n)}(\mathbf{x},\mathbf{M},\lambda)$ is convex on its domain and differentiable on $\mathscr{E}_n^{\succ} \times \mathbb{R}^{\succ}$.

Also, if the body is *homogeneous* we assume that *both $\omega \equiv \omega_{\mathrm{o}}$ and $\Phi^{(n)}$ are independent of* $\mathbf{x}$. Here, and in the sequel, $\mathrm{cof}\,\mathbf{F} \in \mathrm{Lin}_n^{\succ}$ denotes the tensor of cofactors of $\mathbf{F} \in \mathrm{Lin}_n^{\succ}$; thus,

$$\mathrm{cof}\,\mathbf{F} = (\det\mathbf{F})\mathbf{F}^{-\mathrm{T}} \quad \text{for all } \mathbf{F} \in \mathrm{Lin}_n^{\succ}.$$

In general, invariance under a change in observer, (2.1), *does **not** imply* that the function $\Phi^{(n)}$ must satisfy[10]

$$\Phi^{(n)}\big(\mathbf{x},\mathbf{QM},\lambda\big) = \Phi^{(n)}\big(\mathbf{x},\mathbf{M},\lambda\big) \quad \text{for every } \mathbf{M} \in \mathscr{E}_n^{\succ}, \mathbf{Q} \in \mathrm{Orth}_n^{\succ}, \lambda \in \mathbb{R}^{\succ}, \tag{3.2}$$

and $a.e.$ $\mathbf{x} \in \mathscr{B}$. In fact, $\Phi^{(n)}$ *is not uniquely determined* by (3.1) (see, e.g., [18, p. 158]). A particular choice of $\Phi^{(n)}$ may satisfy (3.2), while another does not. However, a recent result of Šilhavý [49] identifies a particular $\Phi^{(n)}$ that satisfies (3.2). In §5.2 we will have occasion to require that this $\Phi^{(n)}$ be used in (3.1).

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[9] This terminology for (3.1) has previously been used in [52].

[10] For $\mathbf{K} \in \mathrm{Lin}$ and $\mathbf{M} = (\mathbf{F},\mathbf{A}) \in \mathscr{E}_3$ we write $\mathbf{KM} := (\mathbf{KF},\mathbf{KA})$.

Let $n = 2$. For such stored-energy functions the Piola-Kirchhoff stress is given by

$$\mathbf{S}(\mathbf{x},\mathbf{F}) = \omega(\mathbf{x})|\mathbf{F}|^{p-2}\mathbf{F} + \mathbf{B}(\mathbf{x},\mathbf{F}) + \Lambda(\mathbf{x},\mathbf{F})\operatorname{cof}\mathbf{F}$$

for *a.e.* $\mathbf{x} \in \mathscr{B}$ and every $\mathbf{F} \in \mathrm{Lin}_2^{\succ}$, where

$$\mathbf{B}(\mathbf{x},\mathbf{F}) := \frac{\partial \Phi^{(2)}(\mathbf{x},\mathbf{F},\lambda)}{\partial \mathbf{F}}\bigg|_{\lambda=\det\mathbf{F}}, \qquad \Lambda(\mathbf{x},\mathbf{F}) := \frac{\partial \Phi^{(2)}(\mathbf{x},\mathbf{F},\lambda)}{\partial \lambda}\bigg|_{\lambda=\det\mathbf{F}}. \tag{3.3}$$

Let $n = 3$. For stored-energy functions that satisfy $(3.1)_2$ it follows that the Piola-Kirchhoff stress satisfies, for *a.e.* $\mathbf{x} \in \mathscr{B}$, every $\mathbf{F} \in \mathrm{Lin}_3^{\succ}$, and every $\mathbf{H} \in \mathrm{Lin}_3$,

$$\mathbf{S}(\mathbf{x},\mathbf{F}) : \mathbf{H} = \mathbf{H} : \Big[\omega(\mathbf{x})|\mathbf{F}|^{p-2}\mathbf{F} + \mathbf{B}(\mathbf{x},\mathbf{F}) + \Lambda(\mathbf{x},\mathbf{F})\operatorname{cof}\mathbf{F}\Big] + \mathbf{D}(\mathbf{x},\mathbf{F}) : \mathbb{K}(\mathbf{F})[\mathbf{H}], \tag{3.4}$$

where

$$\mathbf{B}(\mathbf{x},\mathbf{F}) := \frac{\partial \Phi^{(3)}(\mathbf{x},\mathbf{F},\mathbf{A},\lambda)}{\partial \mathbf{F}}\bigg|_{\substack{\mathbf{A}=\operatorname{cof}\mathbf{F}\\ \lambda=\det\mathbf{F}}}, \qquad \mathbf{D}(\mathbf{x},\mathbf{F}) := \frac{\partial \Phi^{(3)}(\mathbf{x},\mathbf{F},\mathbf{A},\det\mathbf{F})}{\partial \mathbf{A}}\bigg|_{\mathbf{A}=\operatorname{cof}\mathbf{F}},$$
$$\Lambda(\mathbf{x},\mathbf{F}) := \frac{\partial \Phi^{(3)}(\mathbf{x},\mathbf{F},\operatorname{cof}\mathbf{F},\lambda)}{\partial \lambda}\bigg|_{\lambda=\det\mathbf{F}}, \quad \mathbb{K}(\mathbf{F})[\mathbf{H}] := \frac{\mathrm{d}(\operatorname{cof}\mathbf{F})}{\mathrm{d}\mathbf{F}}[\mathbf{H}]. \tag{3.5}$$

4 Uniqueness of Minimizers

4.1 Equilibrium Solutions

In this subsection we consider the displacement, traction, and mixed problems and obtain a uniqueness result that is valid for all of them. For the pure-displacement problem $\mathscr{D} = \partial\mathscr{B}$, $\mathrm{Var} = W_0^{1,n}(\mathscr{B};\mathbb{R}^n) \cap C(\overline{\mathscr{B}};\mathbb{R}^n)$, and we have the following identities (see, e.g., [3] or [42, pp. 28–31]), for all $\mathbf{z} \in W^{1,n}(\mathscr{B};\mathbb{R}^n)$ and $\mathbf{w} \in W_0^{1,n}(\mathscr{B};\mathbb{R}^n)$,

$$\int_{\mathscr{B}} \nabla\mathbf{z} : \operatorname{cof}\nabla\mathbf{w}\,\mathrm{d}\mathbf{x} = 0, \qquad \int_{\mathscr{B}} \det\nabla\mathbf{w}\,\mathrm{d}\mathbf{x} = 0. \tag{4.1}$$

For the mixed and traction problems $(4.1)_1$ is satisfied by all $\mathbf{w} \in \mathrm{Var}$ and $\mathbf{z} \in \mathrm{Trac_o}$, where[11]

$$\mathrm{Trac_o} := \begin{cases} \{\mathbf{z} \in W^{1,n}(\mathscr{B};\mathbb{R}^n) :\ \mathbf{z} = \mathbf{0} \text{ on } \mathscr{S}\} & \text{if } \mathscr{S} \neq \varnothing, \\ W^{1,n}(\mathscr{B};\mathbb{R}^n) & \text{if } \mathscr{D} = \partial\mathscr{B}, \end{cases}$$

i.e., those mappings that are equal to zero on the portion of the boundary where dead-load tractions are prescribed.

Lemma 4.1 *Assume that W is uniformly polyconvex. Let $\mathbf{u}_\mathrm{e}$ be a weak equilibrium solution. Then, for any $\mathbf{v} \in \mathscr{A}$, $\mathbf{z} \in \mathrm{Trac_o}$, $\sigma \in L^\infty(\mathscr{B};[0,1])$, and $\nu \in \mathbb{R}$ ($\nu = 0$ if $\mathscr{D} \neq \partial\mathscr{B}$)*

$$\begin{aligned} E(\mathbf{v}) \geq{} & E(\mathbf{u}_\mathrm{e}) + \int_{\mathscr{B}} \omega(\mathbf{x})\Big(\frac{[1-\sigma(\mathbf{x})]}{2}|\nabla\mathbf{u}_\mathrm{e}(\mathbf{x})|^{p-2}|\nabla\mathbf{w}(\mathbf{x})|^2 + \frac{\kappa_p}{p}\sigma(\mathbf{x})|\nabla\mathbf{w}(\mathbf{x})|^p\Big)\mathrm{d}\mathbf{x} \\ & + \int_{\mathscr{B}} \Big[\Lambda\big(\mathbf{x},\nabla\mathbf{u}_\mathrm{e}(\mathbf{x})\big) - \nu\Big]\det\nabla\mathbf{w}(\mathbf{x})\,\mathrm{d}\mathbf{x} + \delta_{n3}\int_{\mathscr{B}} \mathbf{X}_{\mathrm{e},\mathbf{z}}(\mathbf{x}) : \operatorname{cof}\nabla\mathbf{w}(\mathbf{x})\,\mathrm{d}\mathbf{x}, \end{aligned} \tag{4.2}$$

[11] The equality on the boundary is to be taken in the sense of trace.

where $\mathbf{w} := \mathbf{v} - \mathbf{u}_{\mathrm{e}}$*,* $\kappa_p > 0$ *is given by Proposition A.1, and*

$$\mathbf{X}_{\mathrm{e},\mathbf{z}}(\mathbf{x}) := \mathbf{D}\big(\mathbf{x}, \nabla\mathbf{u}_{\mathrm{e}}(\mathbf{x})\big) + \Lambda\left(\mathbf{x}, \nabla\mathbf{u}_{\mathrm{e}}(\mathbf{x})\right)\nabla\mathbf{u}_{\mathrm{e}}(\mathbf{x}) - \nabla\mathbf{z}(\mathbf{x}). \tag{4.3}$$

Proof We prove the result when $n = 3$. The proof for $n = 2$ is similar. Suppose that W is uniformly polyconvex. Let $\mathbf{F}, \mathbf{G} \in \mathrm{Lin}_3^{\succ}$ and define $\mathbf{H} := \mathbf{G} - \mathbf{F}$. For clarity of exposition we suppress the $\mathbf{x}$ in our calculation. We note that the convexity of $\Phi^{(3)}$ yields (for *a.e.* $\mathbf{x} \in \mathscr{B}$)

$$\widehat{\Phi}^{(3)}(\mathbf{G}) \geq \widehat{\Phi}^{(3)}(\mathbf{F}) + \mathbf{B}(\mathbf{F}) : \mathbf{H} + \mathbf{D}(\mathbf{F}) : \big[\operatorname{cof}\mathbf{G} - \operatorname{cof}\mathbf{F}\big] + \Lambda(\mathbf{F})\big[\det\mathbf{G} - \det\mathbf{F}\big], \tag{4.4}$$

where $\widehat{\Phi}^{(3)}(\mathbf{G}) := \Phi^{(3)}(\mathbf{G}, \operatorname{cof}\mathbf{G}, \det\mathbf{G})$. If we now multiply (A.1), with $\mathbf{a} = \mathbf{G}$ and $\mathbf{b} = \mathbf{F}$, by ω/p and add the result to (4.4) we find, with the aid of $(3.1)_2$, (3.4), (B.2), and (B.3), that, for any $\sigma \in [0,1]$,

$$\begin{aligned} W(\mathbf{G}) \geq W(\mathbf{F}) + \mathbf{S}(\mathbf{F}) : \mathbf{H} + \omega\left(\frac{1}{2}\big[1-\sigma\big]|\mathbf{F}|^{p-2}|\mathbf{H}|^2 + \frac{\kappa_p}{p}\sigma|\mathbf{H}|^p\right) \\ + \mathbf{D}(\mathbf{F}) : \operatorname{cof}\mathbf{H} + \Lambda(\mathbf{F})\big(\det\mathbf{H} + \mathbf{F} : \operatorname{cof}\mathbf{H}\big). \end{aligned} \tag{4.5}$$

Next, let $\mathbf{u}_{\mathrm{e}}$ be a weak equilibrium solution, $\mathbf{v} \in \mathscr{A}$, and define $\mathbf{w} := \mathbf{v} - \mathbf{u}_{\mathrm{e}}$. Suppose that $\sigma \in L^\infty(\mathscr{B};[0,1])$. If we now take $\mathbf{G} = \nabla\mathbf{v}(\mathbf{x})$, $\mathbf{F} = \nabla\mathbf{u}_{\mathrm{e}}(\mathbf{x})$, and $\mathbf{H} = \nabla\mathbf{w}(\mathbf{x})$ in (4.5) and then integrate the result over $\mathscr{B}$ and subtract $(4.1)_1$, we conclude, with the aid of (2.2), (2.3), and (4.3), that (4.2) is satisfied with $\nu = 0$. Finally, if $\mathscr{D} = \partial\mathscr{B}$ then (4.2) follows upon subtracting ν times $(4.1)_2$ from (4.2) with $\nu = 0$. □

We now make use of Lemma 4.1 in order to establish the uniqueness of an energy minimizer subject to certain constraints. We first recall that a mapping $\mathbf{u}_{\mathrm{e}} \in W^{1,n}(\mathscr{B};\mathbb{R}^n) \cap C(\overline{\mathscr{B}};\mathbb{R}^n)$ is a *weak equilibrium solution* if $\det\nabla\mathbf{u}_{\mathrm{e}} > 0$ *a.e.*, $\mathbf{u}_{\mathrm{e}}$ is a weak solution of the equations of equilibrium (2.4) (see (2.3) on p. 6), and $\mathbf{u}_{\mathrm{e}}$ satisfies $\mathbf{u}_{\mathrm{e}} = \mathbf{d}$ on $\mathscr{D}$.

Theorem 4.2 (Uniqueness of Energy Minimizers) *Assume that W is uniformly polyconvex. Let* $\mathbf{u}_{\mathrm{e}}$ *be a weak equilibrium solution.*

(a) If $n = 2$ *suppose that* $\mathbf{u}_{\mathrm{e}}$ *satisfies, for some* $\nu \in \mathbb{R}$ *(*$\nu = 0$ *if* $\mathscr{D} \neq \partial\mathscr{B}$*) and a.e.* $\mathbf{x} \in \mathscr{B}$*,*

$$\big|\Lambda\left(\mathbf{x}, \nabla\mathbf{u}_{\mathrm{e}}(\mathbf{x})\right) - \nu\big| \leq \omega(\mathbf{x})\big|\nabla\mathbf{u}_{\mathrm{e}}(\mathbf{x})\big|^{p-2}. \tag{4.6}$$

(b) If $n = 3$ *suppose that* $\mathbf{u}_{\mathrm{e}}$ *satisfies, for some* $\mathbf{z} \in \mathrm{Trac}_{\mathrm{o}}$*,* $\nu \in \mathbb{R}$ *(*$\nu = 0$ *if* $\mathscr{D} \neq \partial\mathscr{B}$*) and a.e.* $\mathbf{x} \in \mathscr{B}$*,*

$$2\Gamma_{\mathrm{e},\mathbf{z}}(\mathbf{x}) + \beta_p\big|\Lambda\left(\mathbf{x}, \nabla\mathbf{u}_{\mathrm{e}}(\mathbf{x})\right) - \nu\big||\nabla\mathbf{u}_{\mathrm{e}}(\mathbf{x})| \leq \omega(\mathbf{x})|\nabla\mathbf{u}_{\mathrm{e}}(\mathbf{x})|^{p-2}. \tag{4.7}$$

Then $\mathbf{u}_{\mathrm{e}}$ *is an absolute minimizer of E. Moreover, if, in addition,* (4.6) *or* (4.7) *is a strict inequality almost everywhere, then* $\mathbf{u}_{\mathrm{e}}$ *is the unique absolute minimizer of E. Further, there are no other weak equilibrium solutions that satisfy* (4.6) *or* (4.7) *with strict inequality almost everywhere. Here* $\Gamma_{\mathrm{e},\mathbf{z}}$ *is the largest principal stretch of* $\mathbf{X}_{\mathrm{e},\mathbf{z}}$ *given by* (4.3)*,* κ_p *is given by Proposition A.1, and*

$$\beta_p := \frac{2}{3^{3/2}}\left[\frac{p}{2\kappa_p}\right]^{1/(p-2)}. \tag{4.8}$$

Remark 4.3 We observe that Theorem 4.2 does not require the body to be either homogeneous or isotropic. In particular, (4.6) and (4.7) apply to heterogeneous materials such as composites. In Section §6.3 we construct an example, in 2-dimensions, of a simple composite that has a unique solution for certain boundary values.

Remark 4.4 It is not evident from our proof that the failure of (4.6) or (4.7) necessitates the existence of more than one absolute minimizer of E. However, the example in §6.2 indicates that the failure of (4.6) might lead to multiple equilibrium solutions. See Remark 6.4. In addition, in any problem in which one can show that two weak solutions of the equilibrium equations, which have the same energy, do indeed exist, one can immediately conclude that both solutions fail to satisfy (4.6) or (4.7). See Remark 6.1.

Proof We will prove the result for the pure-displacement problem. The result for the mixed and traction problems will follow from the same calculations with $\nu = 0$. We first show that $E(\mathbf{v}) \geq E(\mathbf{u}_e)$ for all $\mathbf{v} \in \mathscr{A}$. Fix $\mathbf{z} \in \mathrm{Trac}_o$ and $\nu \in \mathbb{R}$. We first note that, in view of Lemma 4.1, it suffices to show that there exists a measurable function $\sigma : \mathscr{B} \to [0,1]$ such that, for every $\mathbf{H} \in \mathrm{Lin}_n$ and *a.e.* $\mathbf{x} \in \mathscr{B}$,

$$\omega|\mathbf{H}|^2\left[\frac{(1-\sigma)}{2}|\nabla\mathbf{u}_e|^{p-2} + \frac{\kappa_p\sigma}{p}|\mathbf{H}|^{p-2}\right] + (\Lambda - \nu)\det\mathbf{H} + \delta_{n3}\Big(\mathbf{X}_{e,\mathbf{z}} : \mathrm{cof}\,\mathbf{H}\Big) \geq 0. \quad (4.9)$$

If $n = 2$ then (4.9) follows from (4.6), Hadamard's inequality:

$$2|\det\mathbf{H}| \leq |\mathbf{H}|^2,$$

and the choice $\sigma \equiv 0$. If $n = p = 3$ then (4.9) follows from (4.7), (4.8), Hadamard's inequality (see, e.g., [53, p. 408]):

$$3^{3/2}|\det\mathbf{H}| \leq |\mathbf{H}|^3,$$

the cofactor inequality (B.5), and the choice

$$\sigma(\mathbf{x}) = \frac{\big|\Lambda\big(\mathbf{x}, \nabla\mathbf{u}_e(\mathbf{x})\big) - \nu\big|}{\omega(\mathbf{x})\kappa_3\sqrt{3}}.$$

Now assume that $n = 3$ and $p > 3$. Then Hadamard's inequality and (B.5) reduce (4.9) to showing that there exists a measurable function $\sigma : \mathscr{B} \to [0,1]$ such that, for every $\mathbf{H} \in \mathrm{Lin}_3$ and *a.e.* $\mathbf{x} \in \mathscr{B}$,

$$\omega\left[\frac{(1-\sigma)}{2}|\nabla\mathbf{u}_e|^{p-2} + \frac{\kappa_p\sigma}{p}|\mathbf{H}|^{p-2}\right] - \frac{|\Lambda - \nu|}{3^{3/2}}|\mathbf{H}| - \Gamma_{e,\mathbf{z}} \geq 0, \quad (4.10)$$

where $\Gamma_{e,\mathbf{z}} \geq 0$ is the largest principal stretch of $\mathbf{X}_{e,\mathbf{z}}$ given by (4.3).

Before we determine σ such that (4.7) implies (4.10), we first consider the implications of $\sigma(\mathbf{x}_o) = 0$ at some $\mathbf{x}_o \in \mathscr{B}$. We note that (4.10) with $\sigma(\mathbf{x}_o) = 0$ is satisfied for every $\mathbf{H} \in \mathrm{Lin}_3$ if and only if

$$\big|\Lambda\big(\mathbf{x}_o, \nabla\mathbf{u}_e(\mathbf{x}_o)\big) - \nu\big| = 0 \qquad \text{and} \qquad 2\Gamma_{e,\mathbf{z}}(\mathbf{x}_o) \leq \omega(\mathbf{x}_o)|\nabla\mathbf{u}_e(\mathbf{x}_o)|^{p-2}.$$

We therefore conclude that:

(i) If $|\Lambda(\mathbf{x}_o, \nabla\mathbf{u}_e(\mathbf{x}_o)) - \nu| = 0$ at some $\mathbf{x}_o \in \mathscr{B}$, then (4.7) yields (4.10) with $\sigma(\mathbf{x}_o) = 0$.
(ii) If $|\Lambda(\mathbf{x}_o, \nabla\mathbf{u}_e(\mathbf{x}_o)) - \nu| \neq 0$ for some $\mathbf{x}_o \in \mathscr{B}$, then $\sigma(\mathbf{x}_o) \neq 0$.

Next, since $\omega(\mathbf{x}) \geq \omega_o > 0$ and $\det\nabla\mathbf{u}_e > 0$ *a.e.*, we can fix $\mathbf{x}_o \in \mathscr{B}$ and assume that

$$\omega(\mathbf{x}_o)|\nabla\mathbf{u}_e(\mathbf{x}_o)| > 0, \qquad \big|\Lambda\big(\mathbf{x}_o, \nabla\mathbf{u}_e(\mathbf{x}_o)\big) - \nu\big| > 0, \ \text{ and (hence) } \ \sigma(\mathbf{x}_o) \in (0,1]. \quad (4.11)$$

Define $t := |\mathbf{H}| \geq 0$. Then (4.10) can be viewed as

$$f(t) := at^{p-2} - bt + c \geq 0 \ \text{ for all } t \geq 0, \quad (4.12)$$

where, in view of (4.7), $c = f(0) \geq 0$, $b \geq 0$, and $a > 0$ (since $\sigma > 0$). A necessary and sufficient condition for (4.12) to be satisfied is that f be nonnegative at the unique $t_{\rm m}$ that satisfies $f'(t_{\rm m}) = 0$, i.e., $a(p-2)t_{\rm m}^{p-3} = b$. If we substitute $t_{\rm m}$ into (4.12) we find that (4.10) is a consequence of

$$f(t_{\rm m}) = c - b\left(\frac{p-3}{p-2}\right)\left[\frac{b}{a(p-2)}\right]^{1/(p-3)} \geq 0$$

or, equivalently,

$$2\Gamma_{\rm e,\mathbf{z}} + \delta_p\left[\frac{|\Lambda - \nu|^{p-2}}{\omega\sigma}\right]^{1/(p-3)} \leq \omega|\nabla\mathbf{u}_{\rm e}|^{p-2}(1-\sigma), \tag{4.13}$$

where

$$\delta_p := \frac{2(p-3)}{\left[3^{3/2}(p-2)\right]^{(p-2)/(p-3)}}\left[\frac{p}{\kappa_p}\right]^{1/(p-3)}. \tag{4.14}$$

Next, define

$$\sigma := \frac{\omega|\nabla\mathbf{u}_{\rm e}|^{p-2} - 2\Gamma_{\rm e,\mathbf{z}}}{(p-2)\omega|\nabla\mathbf{u}_{\rm e}|^{p-2}} > 0, \qquad 1-\sigma = \frac{2\Gamma_{\rm e,\mathbf{z}} + (p-3)\omega|\nabla\mathbf{u}_{\rm e}|^{p-2}}{(p-2)\omega|\nabla\mathbf{u}_{\rm e}|^{p-2}} > 0,$$

where $\sigma \in (0,1)$ follow from $\Gamma_{\rm e,\mathbf{z}} \geq 0$, (4.7), and (4.11). With this choice of σ inequality (4.13) becomes

$$2\Gamma_{\rm e,\mathbf{z}} + \delta_p\left[\frac{(p-2)|\nabla\mathbf{u}_{\rm e}|^{p-2}|\Lambda-\nu|^{p-2}}{\omega|\nabla\mathbf{u}_{\rm e}|^{p-2} - 2\Gamma_{\rm e,\mathbf{z}}}\right]^{1/(p-3)} \leq \frac{2\Gamma_{\rm e,\mathbf{z}} + (p-3)\omega|\nabla\mathbf{u}_{\rm e}|^{p-2}}{(p-2)},$$

which, after some algebra, reduces to

$$\left[\delta_p\frac{(p-2)}{(p-3)}\right]^{(p-3)/(p-2)}(p-2)^{1/(p-2)}|\nabla\mathbf{u}_{\rm e}||\Lambda-\nu| \leq \omega|\nabla\mathbf{u}_{\rm e}|^{p-2} - 2\Gamma_{\rm e,\mathbf{z}}. \tag{4.15}$$

However, (4.8) and (4.14) yield

$$(p-2)^{1/(p-2)}\left[\delta_p\frac{(p-2)}{(p-3)}\right]^{(p-3)/(p-2)} = \beta_p$$

which shows that (4.15) and (4.7) are identical.

We next note that it is clear that if (4.6) or (4.7) is a strict inequality almost everywhere then (4.2) and the above proof yield $E(\mathbf{v}) > E(\mathbf{u}_{\rm e})$ unless $\nabla\mathbf{v} = \nabla\mathbf{u}_{\rm e}$ *a.e.* Since $\mathscr{B}$ is open and connected it follows that $\mathbf{v} = \mathbf{u}_{\rm e} + \mathbf{a}$ *a.e.* for some $\mathbf{a} \in \mathbb{R}^n$. If $\mathscr{D} \neq \varnothing$, then $\mathbf{u}_{\rm e} = \mathbf{v} = \mathbf{d}$ on the nonempty, relatively open set $\mathscr{D}$, while if $\mathscr{D} = \varnothing$, then (see (2.7))

$$\int_{\mathscr{B}}(\mathbf{u}_{\rm e} - \mathbf{v})\,\mathrm{d}\mathbf{x} = \mathbf{0}.$$

In either case $\mathbf{a} = \mathbf{0}$.

Finally, if we suppose that $\mathbf{v}_{\rm e} \not\equiv \mathbf{u}_{\rm e}$ is a weak equilibrium solution that satisfies (4.6) or (4.7) with strict inequality almost everywhere (and with $\mathbf{u}_{\rm e}$ replaced by $\mathbf{v}_{\rm e}$), then the above argument yields $E(\mathbf{u}_{\rm e}) > E(\mathbf{v}_{\rm e})$, which is a contradiction. □

Remark 4.5 Suppose we replace the assumption that $(\mathbf{M},\lambda)\mapsto\Phi^{(n)}(\mathbf{x},\mathbf{M},\lambda)$ is (globally) convex with the weaker assumption[12] that, for *a.e.* $\mathbf{x}\in\mathscr{B}$,

$$\big(\mathbf{M}_{\rm e}(\mathbf{x}),\det\nabla\mathbf{u}_{\rm e}(\mathbf{x})\big)\ \textit{is a point of convexity of}\ (\mathbf{M},\lambda)\mapsto\Phi^{(n)}(\mathbf{x},\mathbf{M},\lambda), \tag{4.16}$$

where $\mathbf{M}_{\rm e}(\mathbf{x})=\nabla\mathbf{u}_{\rm e}(\mathbf{x})$ if $n=2$ or $\mathbf{M}_{\rm e}(\mathbf{x})=(\nabla\mathbf{u}_{\rm e}(\mathbf{x}),\operatorname{cof}\nabla\mathbf{u}_{\rm e}(\mathbf{x}))$ if $n=3$. Then it is clear from the proof of Theorem 4.2 that any weak equilibrium solution $\mathbf{u}_{\rm e}$ that satisfies (4.16) and either (4.6) or (4.7) is an absolute minimizer of E. Theorem 4.2 therefore has implications for stored-energies that admit phase transitions (see Ball and James [6] or, e.g., Grabovsky and Truskinovsky [27] and the references therein); while one would not expect uniqueness of minimizers for such constitutive relations, Theorem 4.2 yields conditions under which an equilibrium solution is an absolute minimizer of the energy.

Remark 4.6 (a) Suppose that $p=n$ and, for *a.e.* $\mathbf{x}$,

$$|\Lambda(\mathbf{x},\mathbf{F})|\geq\varphi(\mathbf{F}),\ \text{ where }\quad\varphi(\mathbf{F})\to\infty\ \text{ as }\ |\mathbf{F}|\to\infty. \tag{4.17}$$

Then any admissible deformation $\mathbf{u}\in\mathscr{A}$ that satisfies (4.6) or (4.7) must have additional regularity, i.e., $\mathbf{u}\in W^{1,\infty}(\mathscr{B};\mathbb{R}^n)$.

(b) Suppose that $p\geq n$ and, for *a.e.* $\mathbf{x}$,

$$|\Lambda(\mathbf{x},\mathbf{F})|\geq\varphi(\mathbf{F}),\ \text{ where }\quad\varphi(\mathbf{F})\to\infty\ \text{ as }\ \det\mathbf{F}\to 0^+. \tag{4.18}$$

Then any $\mathbf{u}\in\mathscr{A}$ that satisfies (4.6) or (4.7) will satisfy, for some $\varepsilon>0$,

$$\det\nabla\mathbf{u}(\mathbf{x})>\varepsilon\quad\text{for } a.e.\ \mathbf{x}\in\mathscr{B}.$$

The next result, which is an immediate consequence of Remark 2.7 and Theorem 4.2, shows that, for the genuine mixed problem on a homogeneous body with $\Phi^{(n)}$ continuously differentiable, the set of homogeneous deformations that satisfies (4.7) with strict inequality and $\mathbf{z}\equiv\mathbf{0}$ or (4.6) with strict inequality is an open subset of $\mathrm{Lin}_n^{\succ}$.

Proposition 4.7 *Let both $\mathscr{D}$ and $\mathscr{S}$ be nonempty. Assume that the body is homogeneous and that W is uniformly polyconvex with $\Phi^{(n)}$ continuously differentiable.*

(a) If $n=2$ suppose that $\mathbf{F}_{\rm e}\in\mathrm{Lin}_2^{\succ}$ satisfies

$$|\Lambda(\mathbf{F}_{\rm e})|<\omega_{\rm o}|\mathbf{F}_{\rm e}|^{p-2}. \tag{4.19}$$

(b) If $n=3$ suppose that $\mathbf{F}_{\rm e}\in\mathrm{Lin}_3^{\succ}$ satisfies

$$2\Gamma_{\rm e}+\beta_p|\Lambda(\mathbf{F}_{\rm e})||\mathbf{F}_{\rm e}|<\omega_{\rm o}|\mathbf{F}_{\rm e}|^{p-2} \tag{4.20}$$

where $\Gamma_{\rm e}$ is the largest principal stretch of $\mathbf{X}_{\rm e}:=\mathbf{D}(\mathbf{F}_{\rm e})+\Lambda(\mathbf{F}_{\rm e})\mathbf{F}_{\rm e}$.

Suppose further that $\mathbf{b}\equiv\mathbf{0}$, $\mathbf{d}$ is given by $\mathbf{d}(\mathbf{x}):=\mathbf{F}_{\rm e}\mathbf{x}+\mathbf{a}$ for some $\mathbf{a}\in\mathbb{R}^n$ and all $\mathbf{x}\in\mathscr{D}$, and $\mathbf{s}$ is given by $\mathbf{s}(\mathbf{x}):=\mathbf{S}(\mathbf{F}_{\rm e})\mathbf{n}(\mathbf{x})$ for $\mathbf{x}\in\mathscr{S}$, where $\mathbf{n}(\mathbf{x})$ denotes the outward unit normal to the boundary (for $\mathscr{H}^{n-1}$-a.e. $\mathbf{x}\in\mathscr{S}$). Then the classical equilibrium solution $\mathbf{u}_{\rm e}(\mathbf{x}):=\mathbf{F}_{\rm e}\mathbf{x}+\mathbf{a}$ is the unique absolute minimizer of the energy E. Moreover, there does not exist a weak equilibrium solution $\mathbf{v}_{\rm e}\not\equiv\mathbf{u}_{\rm e}$ that satisfies (4.19) *or* (4.20) *with $\mathbf{F}_{\rm e}$ replaced by $\nabla\mathbf{v}_{\rm e}(\mathbf{x})$.*

[12] We say that $\mathbf{A}$ is *a point of convexity* of the differentiable, real-valued function $\mathbf{X}\mapsto\phi(\mathbf{X})$ whenever the graph of ϕ is (globally) above its tangent plane at $\mathbf{A}$, i.e., $\phi(\mathbf{X})\geq\phi(\mathbf{A})+\nabla\phi(\mathbf{A})\cdot(\mathbf{X}-\mathbf{A})$ for all $\mathbf{X}$.

Remark 4.8 Proposition 4.7 can be viewed as a simple analogue, for the mixed problem, of the well-known result of Knops and Stuart [37]. Consider a homogeneous star-shaped body $\mathscr{T}$. Fix $\mathbf{F}_\mathrm{e} \in \mathrm{Lin}_n^{\succ}$ and consider the pure-displacement problem: $\mathscr{S} = \varnothing$ with $\mathrm{d}(\mathbf{x}) := \mathbf{F}_\mathrm{e}\mathbf{x}$ for $\mathbf{x} \in \partial\mathscr{T}$. Then results in [37] show that, in the absence of body forces, there is at most one deformation $\mathbf{u}_\mathrm{e} \in C^2(\mathscr{T};\mathbb{R}^n) \cap C^1(\overline{\mathscr{T}};\mathbb{R}^n) \cap \mathscr{A}$ that satisfies the equilibrium equations (2.4) *provided* that the stored-energy function W is strictly quasiconvex at $\mathbf{F}_\mathrm{e}$ and (globally) rank-one convex. Similarly, Bevan [9] has shown that there is at most one weak relative minimizer $\mathbf{u}_\mathrm{m} \in C^1(\overline{\mathscr{T}};\mathbb{R}^n) \cap \mathscr{A}$ for certain stored-energy functions $W(\mathbf{F}) = \Psi(\mathbf{F}) + h(\det \mathbf{F})$ with h convex, Ψ strictly quasiconvex, and both functions having appropriate growth.

4.2 Weak Relative Minimizers

We now assume that we are given an admissible deformation $\mathbf{u}_\mathrm{m}$ that is a weak relative minimizer (see Definition 2.8) rather than an equilibrium solution. Then the additional hypotheses of Lemma 2.9 allow us to conclude that $\mathbf{u}_\mathrm{m}$ is a weak equilibrium solution and so we can then apply the results from the previous subsection. The next result follows directly from Lemma 2.9 and Theorem 4.2.

Proposition 4.9 *Let $\mathbf{u}_\mathrm{m} \in \mathscr{A} \cap W^{1,\infty}(\mathscr{B};\mathbb{R}^n)$ be a weak relative minimizer of E that satisfies* (2.8) *and either* (4.6) *or* (4.7) *with strict inequality on a set of positive measure. Suppose, in addition, that* $\mathbf{S}$ *satisfies* (2.9). *Then $\mathbf{u}_\mathrm{m}$ is a weak equilibrium solution that is the unique absolute minimizer of E.*

Remark 4.6 together with the above proposition then gives us the following result.

Corollary 4.10 *Assume that $p = n$. Let $\mathbf{u}_\mathrm{m} \in \mathscr{A}$ be a weak relative minimizer of E that satisfies* (4.6) *or* (4.7) *with strict inequality on a set of positive measure. Suppose, in addition, that* $\mathbf{S}$ *satisfies* (2.9) *and that Λ satisfies* (4.17) *and* (4.18). *Then $\mathbf{u}_\mathrm{m}$ is a weak equilibrium solution that is the unique absolute minimizer of E.*

5 Further results for the Displacement Problem: a Theorem of Zhang and a Theorem of John

5.1 A Theorem of Zhang

We now present a result of Zhang [63] who showed that, in 3-dimensions for the pure-displacement problem, there is at most one equilibrium solution $\mathbf{u}_\mathrm{e}$ that is uniformly close, in $W^{1,\infty}$, to a given homogeneous deformation and, moreover, that $\mathbf{u}_\mathrm{e}$ must then be the minimizer of the energy obtained by both the direct method of the calculus of variations and the implicit function theorem.

Theorem 5.1 (Zhang [63]) *Suppose $\mathscr{D} = \partial\mathscr{B}$ and that W is homogeneous and uniformly polyconvex.*[13] *Fix $\mathbf{F}_\mathrm{o} \in \mathrm{Lin}^{\succ}$. Assume that $(\mathbf{M},\lambda) \mapsto \Phi^{(n)}(\mathbf{M},\lambda)$ is continuously differentiable at $(\mathbf{F}_\mathrm{o}, \det \mathbf{F}_\mathrm{o})$ if $n = 2$ or $(\mathbf{F}_\mathrm{o}, \operatorname{cof} \mathbf{F}_\mathrm{o}, \det \mathbf{F}_\mathrm{o})$ if $n = 3$. Then there exists a $\delta = \delta(\mathbf{F}_\mathrm{o}) > 0$ such that any weak equilibrium solution $\mathbf{u}_\mathrm{e}$ that satisfies*

$$\left\| \nabla \mathbf{u}_\mathrm{e} - \mathbf{F}_\mathrm{o} \right\|_{L^\infty(\mathscr{B})} < \delta \tag{5.1}$$

[13] Zhang [63] instead assumes that $W(\mathbf{F}) = a|\mathbf{F}|^p + b|\operatorname{cof}\mathbf{F}|^q + \Phi(\mathbf{F}, \operatorname{cof}\mathbf{F}, \det\mathbf{F})$ with $a > 0$, $b > 0$, $p \geq 2$, $q \geq p/(p-1)$, and Φ convex.

is a strict absolute minimizer of E. Consequently, there is at most one weak equilibrium solution that satisfies (5.1)*.*

Remark 5.2 The main difficulty is in showing that there are *any* weak equilibrium solutions, especially solutions that satisfy (5.1). However, in this instance and with suitable additional assumptions one can make use of the implicit function theorem to get classical solutions of the equations of equilibrium (2.4) (see Zhang [63]). In particular, if one assumes that the boundary is sufficiently smooth, and one replaces $\mathbf{b}$ with $\varepsilon\mathbf{b}$ and the boundary condition $\mathbf{u} = \mathbf{d}$ on $\partial\mathscr{B}$ with

$$\mathbf{u}(\mathbf{x}) = \mathbf{F}_\mathrm{o}\mathbf{x} + \varepsilon\mathbf{d}(\mathbf{x}) \quad \text{for } \mathbf{x} \in \partial\mathscr{B},$$

where $\mathbf{b}$ and $\mathbf{d}$ are sufficiently smooth, then results of Valent [60] (see, also, [14, Chapter 6] or [48, §20.9]) yield the existence of a classical solution of the equilibrium equations for small ε.

Proof of Theorem 5.1 We prove the result when $n = 3$. The proof for $n = 2$ is similar. For clarity of exposition, we suppress the variable $\mathbf{x}$ as well as the "almost every $\mathbf{x}$" that should accompany most of our inequalities. Fix $\mathbf{F}_\mathrm{o} \in \mathrm{Lin}^{\succ}$ and assume the hypotheses of the theorem. We will show that, if $\delta < \frac{1}{2}|\mathbf{F}_\mathrm{o}|$ in (5.1), then the right-hand side of (4.7) is bounded away from zero, while the left-hand side of (4.7) goes to zero as δ approaches zero. This will allow us to apply Theorem 4.2 to obtain the desired uniqueness.

Define

$$\varepsilon := \frac{\omega_\mathrm{o}|\mathbf{F}_\mathrm{o}|^{p-2}}{2^{p-1}\big[(2+\beta_p)|\mathbf{F}_\mathrm{o}|+1\big]}, \tag{5.2}$$

where β_p is given by (4.8). Then in view of the continuity of $\mathbf{D}$ and Λ at $\mathbf{F}_\mathrm{o}$ there exists a $\delta = \delta(\mathbf{F}_\mathrm{o}, \varepsilon) > 0$ such that, for $\mathbf{G} \in \mathrm{Lin}_3^{\succ}$,

$$|\mathbf{G} - \mathbf{F}_\mathrm{o}| < \delta \quad \Longrightarrow \quad |\Lambda(\mathbf{G}) - \Lambda(\mathbf{F}_\mathrm{o})| < \varepsilon, \quad |\mathbf{D}(\mathbf{G}) - \mathbf{D}(\mathbf{F}_\mathrm{o})| < \varepsilon. \tag{5.3}$$

Without loss of generality, we assume that

$$2\delta < |\mathbf{F}_\mathrm{o}|. \tag{5.4}$$

Let $\mathbf{u}_\mathrm{e}$ be a weak equilibrium solution that satisfies (5.1) with δ given in (5.3)–(5.4). We note that by the triangle inequality, (5.1), and (5.4)

$$\begin{aligned} 2|\mathbf{F}_\mathrm{o}| &\le 2|\mathbf{F}_\mathrm{o} - \nabla\mathbf{u}_\mathrm{e}| + 2|\nabla\mathbf{u}_\mathrm{e}| \le |\mathbf{F}_\mathrm{o}| + 2|\nabla\mathbf{u}_\mathrm{e}|, \\ 2|\nabla\mathbf{u}_\mathrm{e}| &\le 2|\nabla\mathbf{u}_\mathrm{e} - \mathbf{F}_\mathrm{o}| + 2|\mathbf{F}_\mathrm{o}| \le 3|\mathbf{F}_\mathrm{o}|, \end{aligned} \tag{5.5}$$

and hence, in particular,

$$\omega_\mathrm{o}|\mathbf{F}_\mathrm{o}|^{p-2} \le \omega_\mathrm{o}2^{p-2}|\nabla\mathbf{u}_\mathrm{e}|^{p-2}. \tag{5.6}$$

Next, by the triangle inequality, (5.1), (5.3) with $\mathbf{G} = \nabla\mathbf{u}_\mathrm{e}$, and (5.5),

$$\begin{aligned} &\Big|\big[\mathbf{D}(\nabla\mathbf{u}_\mathrm{e}) + \Lambda(\nabla\mathbf{u}_\mathrm{e})\nabla\mathbf{u}_\mathrm{e}\big] - \big[\mathbf{D}(\mathbf{F}_\mathrm{o}) + \Lambda(\mathbf{F}_\mathrm{o})\nabla\mathbf{u}_\mathrm{e}\big]\Big| \\ &\le \big|\mathbf{D}(\nabla\mathbf{u}_\mathrm{e}) - \mathbf{D}(\mathbf{F}_\mathrm{o})\big| + \big|[\Lambda(\nabla\mathbf{u}_\mathrm{e}) - \Lambda(\mathbf{F}_\mathrm{o})]\nabla\mathbf{u}_\mathrm{e}\big| < \varepsilon(1 + 2|\mathbf{F}_\mathrm{o}|). \end{aligned} \tag{5.7}$$

Finally, define

$$\nu := \Lambda(\mathbf{F}_\mathrm{o}), \qquad \mathbf{z}(\mathbf{x}) := \mathbf{D}(\mathbf{F}_\mathrm{o})\mathbf{x} + \Lambda(\mathbf{F}_\mathrm{o})\mathbf{u}_\mathrm{e}(\mathbf{x}) \tag{5.8}$$

and note that $(5.7)_1$ is the norm of $\mathbf{X}_{\mathrm{e},\mathbf{z}}$ given by (4.3). Therefore, (5.1), (5.3) with $\mathbf{G} = \nabla\mathbf{u}_\mathrm{e}$, (5.5), (5.7), and (5.8), together with (B.5) ($\Gamma_{\mathrm{e},\mathbf{z}} \le |\mathbf{X}_{\mathrm{e},\mathbf{z}}|$), yield

$$2\Gamma_{\mathrm{e},\mathbf{z}} + \beta_p\big|\Lambda(\nabla\mathbf{u}_\mathrm{e}) - \nu\big||\nabla\mathbf{u}_\mathrm{e}| < 2\varepsilon\big[(2+\beta_p)|\mathbf{F}_\mathrm{o}| + 1\big]. \tag{5.9}$$

The desired uniqueness now follows (5.2), (5.6), (5.9), and Theorem 4.2. □

5.2 A Theorem of John

We next show that our results also imply a result of John [34] who showed that, in 3-dimensions, there is at most one solution of the pure-displacement problem with small *strain* $\mathbf{E} := \frac{1}{2}[(\nabla\mathbf{u})^{\mathrm{T}}\nabla\mathbf{u} - \mathbf{I}]$.

We first recall that Šilhavý [49] identifies a particular $\Phi^{(n)}$ that satisfies (see (3.2))

$$\Phi^{(n)}\big(\mathbf{QM},\lambda\big) = \Phi^{(n)}\big(\mathbf{M},\lambda\big) \quad \text{for every } \mathbf{M} \in \mathscr{E}_n^{\succ},\, \mathbf{Q} \in \mathrm{Orth}_n^{\succ},\, \lambda \in \mathbb{R}^{\succ}, \tag{5.10}$$

when W is homogeneous. Suppose that this $\Phi^{(n)}$ is used in (3.1). It follows from (5.10) that the derivatives of $\Phi^{(n)}$ (see $(3.3)_2$ and $(3.5)_{3,2}$) satisfy, for every $\mathbf{F} \in \mathrm{Lin}_n^{\succ}$ and $\mathbf{Q} \in \mathrm{Orth}_n^{\succ}$,

$$\Lambda(\mathbf{QF}) = \Lambda(\mathbf{F}), \qquad \mathbf{D}(\mathbf{QF}) = \mathbf{QD}(\mathbf{F}).$$

Standard representation theorems (see, e.g., [14, Theorems 3.3-1 and 4.2-1] or [28, §25, §27] and [48, Theorem 8.3.3]) then yield functions[14] $\Lambda^* : \mathrm{Psym}_n \to \mathbb{R}$ and $\mathbf{D}^* : \mathrm{Psym}_3 \to \mathrm{Sym}_3$ that satisfy, for every $\mathbf{F} \in \mathrm{Lin}_n^{\succ}$,

$$\Lambda(\mathbf{F}) = \Lambda^*\big(\mathbf{F}^{\mathrm{T}}\mathbf{F}\big), \qquad \mathbf{D}(\mathbf{F}) = \mathbf{FD}^*\big(\mathbf{F}^{\mathrm{T}}\mathbf{F}\big). \tag{5.11}$$

John's theorem then follows from (5.11) and our proof of Zhang's theorem.

Theorem 5.3 (John [34]) *Let $\mathscr{D} = \partial\mathscr{B}$ and let W be homogeneous and uniformly polyconvex. Suppose that Λ satisfies $(5.11)_1$ with Λ^* continuous at $\mathbf{I}$. If $n = 3$ suppose, in addition, that $\mathbf{D}$ satisfies $(5.11)_2$ and that $\mathbf{D}^*$ is continuous at $\mathbf{I}$ with $\mathbf{D}^*(\mathbf{I}) = \xi\mathbf{I}$ for some $\xi \in \mathbb{R}$. Then there exists a $\delta > 0$ such that there is at most one weak equilibrium solution $\mathbf{u}_{\mathrm{e}}$ that satisfies*

$$\big\|(\nabla\mathbf{u}_{\mathrm{e}})^{\mathrm{T}}\nabla\mathbf{u}_{\mathrm{e}} - \mathbf{I}\big\|_{L^\infty(\mathscr{B})} < \delta. \tag{5.12}$$

Moreover, if such a $\mathbf{u}_{\mathrm{e}}$ exists it is a strict absolute minimizer of E.

Remark 5.4 The corresponding theorem in [34] does not assume polyconvexity.[15] Instead, John assumes that the stored-energy function $W \in C^3(\mathrm{Lin}_3;\mathbb{R})$ satisfies[16]

$$W(\mathbf{F}) = \mu|\mathbf{E}|^2 + \tfrac{1}{2}\lambda(\mathrm{tr}\,\mathbf{E})^2 + O(|\mathbf{E}|^3), \tag{5.13}$$

where $\mu > 0$ and $\lambda > 0$ denote the Lamé moduli and $\mathbf{E} := \frac{1}{2}[\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}]$. The proof in [34] makes use of the properties of BMO developed by John and Nirenberg [35] rather than the elementary techniques used herein.

Remark 5.5 Let $n = 3$ and suppose that W is isotropic. Then results of Šilhavý [49] yield a $\Phi^{(3)}$ that is isotropic. It follows that the corresponding $\mathbf{D}$ is isotropic and hence, by the representation theorem for isotropic tensor-valued functions, $\mathbf{D}(\mathbf{F}) = \phi_1(\mathbf{B})\mathbf{I} + \phi_2(\mathbf{B})\mathbf{B} + \phi_3(\mathbf{B})\mathbf{B}^2$, where $\mathbf{B} := \mathbf{FF}^{\mathrm{T}}$ and $\phi_i : \mathrm{Psym}_3 \to \mathbb{R}$. Thus, $\mathbf{D}^*(\mathbf{I}) = \mathbf{D}(\mathbf{I}) = \xi\mathbf{I}$ follows from isotropy. Whether or not W is isotropic, one can show that the assumption $\mathbf{S}(\mathbf{I}) = \mathbf{0}$, which is implicit in (5.13), yields $\mathbf{B}(\mathbf{I}) - \mathbf{D}(\mathbf{I}) = \eta\mathbf{I}$ for some $\eta \in \mathbb{R}$.

[14] We write Sym_n for those $\mathbf{H} \in \mathrm{Lin}_n$ that satisfy $\mathbf{H} = \mathbf{H}^{\mathrm{T}}$; Psym_n denotes those $\mathbf{H} \in \mathrm{Sym}_n$ that are strictly positive definite.

[15] The proof in [34, Eqns. (8)–(11)] is also not compatible with $W(\mathbf{F}) = +\infty$ when $\det\mathbf{F} = 0$.

[16] More generally, it is not difficult to show that the results in [34] are valid for any $W \in C^3(\mathrm{Lin}_n;\mathbb{R})$ with $\mathbf{S}(\mathbf{I}) = \mathbf{0}$ and $\mathbb{C}(\mathbf{I})$ strongly elliptic, where $\mathbb{C}(\mathbf{F}) := \partial\mathbf{S}/\partial\mathbf{F}$ here denotes the elasticity tensor. In particular, for a W that satisfies (5.13), $\mu > 0$ and $\mu + \lambda > 0$ suffice. See, e.g., [14, Theorem 4.10.2] for examples of polyconvex energies that are consistent with (5.13).

Proof of Theorem 5.3 We prove the result when $n = 3$. The proof for $n = 2$ is similar. Assume the hypotheses of the theorem. We will show that, if $\delta < \frac{1}{2}$ in (5.12), then the right-hand side of (4.7) is bounded away from zero, while the left-hand side of (4.7) goes to zero as δ approaches zero. This will allow us to once again apply Theorem 4.2 to obtain the desired uniqueness.

Define

$$\varepsilon := \frac{\omega_{\mathrm{o}}}{2(4+\beta_p)}, \tag{5.14}$$

where β_p is given by (4.8). Then, in view of the continuity of Λ^* and $\mathbf{D}^*$ at $\mathbf{I}$ and the fact that $\mathbf{D}^*(\mathbf{I}) = \xi\mathbf{I}$, there exists a $\delta = \delta(\varepsilon) > 0$ such that, for $\mathbf{C} \in \mathrm{Psym}_3$,

$$|\mathbf{C}-\mathbf{I}| < \delta \quad \Longrightarrow \quad |\Lambda^*(\mathbf{C}) - \Lambda^*(\mathbf{I})| < \varepsilon, \quad |\mathbf{D}^*(\mathbf{C}) - \xi\mathbf{I}| < \varepsilon. \tag{5.15}$$

Without loss of generality, we assume that

$$2\delta < 1. \tag{5.16}$$

Let $\mathbf{u}_{\mathrm{e}}$ be a weak equilibrium solution that satisfies (5.12) with δ given in (5.15)–(5.16). We note that, by the triangle inequality, (5.12), (5.16), and the Cauchy-Schwarz inequality,

$$\begin{aligned} &2\sqrt{3} = 2|\mathbf{I}| \le 2|\mathbf{I} - (\nabla\mathbf{u}_{\mathrm{e}})^{\mathrm{T}}\nabla\mathbf{u}_{\mathrm{e}}| + 2|(\nabla\mathbf{u}_{\mathrm{e}})^{\mathrm{T}}\nabla\mathbf{u}_{\mathrm{e}}| < 1 + 2|\nabla\mathbf{u}_{\mathrm{e}}|^2, \\ &|\nabla\mathbf{u}_{\mathrm{e}}|^2 = \mathbf{I} : \big[(\nabla\mathbf{u}_{\mathrm{e}})^{\mathrm{T}}\nabla\mathbf{u}_{\mathrm{e}} - \mathbf{I} + \mathbf{I}\big] \le \sqrt{3}\,\big|(\nabla\mathbf{u}_{\mathrm{e}})^{\mathrm{T}}\nabla\mathbf{u}_{\mathrm{e}} - \mathbf{I}\big| + 3 < 4, \end{aligned} \tag{5.17}$$

and hence, in particular,

$$\omega_{\mathrm{o}} < \omega_{\mathrm{o}}|\nabla\mathbf{u}_{\mathrm{e}}|^{p-2}. \tag{5.18}$$

Next, by the triangle inequality, (5.12), (5.15) with $\mathbf{C} = (\nabla\mathbf{u}_{\mathrm{e}})^{\mathrm{T}}\nabla\mathbf{u}_{\mathrm{e}}$, and (5.17),

$$\begin{aligned} &\Big|\big[\nabla\mathbf{u}_{\mathrm{e}}\mathbf{D}^*\big((\nabla\mathbf{u}_{\mathrm{e}})^{\mathrm{T}}\nabla\mathbf{u}_{\mathrm{e}}\big) + \Lambda^*\big((\nabla\mathbf{u}_{\mathrm{e}})^{\mathrm{T}}\nabla\mathbf{u}_{\mathrm{e}}\big)\nabla\mathbf{u}_{\mathrm{e}}\big] - \big[\xi\nabla\mathbf{u}_{\mathrm{e}} + \Lambda^*(\mathbf{I})\nabla\mathbf{u}_{\mathrm{e}}\big]\Big| \\ &\quad \le \Big|\nabla\mathbf{u}_{\mathrm{e}}\big[\mathbf{D}^*\big((\nabla\mathbf{u}_{\mathrm{e}})^{\mathrm{T}}\nabla\mathbf{u}_{\mathrm{e}}\big) - \xi\mathbf{I}\big]\Big| + \Big|\big[\Lambda^*\big((\nabla\mathbf{u}_{\mathrm{e}})^{\mathrm{T}}\nabla\mathbf{u}_{\mathrm{e}}\big) - \Lambda^*(\mathbf{I})\big]\nabla\mathbf{u}_{\mathrm{e}}\Big| < 4\varepsilon. \end{aligned} \tag{5.19}$$

Finally, define

$$\nu := \Lambda^*(\mathbf{I}), \qquad \mathbf{z}(\mathbf{x}) := \big[\xi + \Lambda^*(\mathbf{I})\big]\mathbf{u}_{\mathrm{e}}(\mathbf{x}) \tag{5.20}$$

and note that, in view of (5.11) and the fact that $\mathbf{D}^*(\mathbf{I}) = \xi\mathbf{I}$, $(5.19)_1$ is the norm of $\mathbf{X}_{\mathrm{e},\mathbf{z}}$ given by (4.3). Then (5.11), (5.12), (5.15) with $\mathbf{C} = (\nabla\mathbf{u}_{\mathrm{e}})^{\mathrm{T}}\nabla\mathbf{u}_{\mathrm{e}}$, (5.17), (5.19), and (5.20), together with (B.5) ($\Gamma_{\mathrm{e},\mathbf{z}} \le |\mathbf{X}_{\mathrm{e},\mathbf{z}}|$), yield

$$2\Gamma_{\mathrm{e},\mathbf{z}} + \beta_p\big|\Lambda(\nabla\mathbf{u}_{\mathrm{e}}) - \nu\big||\nabla\mathbf{u}_{\mathrm{e}}| < 2\varepsilon(4+\beta_p). \tag{5.21}$$

The desired uniqueness now follows (5.14), (5.18), (5.21), and Theorem 4.2. □

6 Examples

6.1 An Application of Theorem 5.1 to a Thick Spherical Shell or an Annulus

Fix $b > a > 0$ and let $\mathscr{B} := \{\mathbf{x} \in \mathbb{R}^n : a < |\mathbf{x}| < b\}$, where $n = 2$ or $n = 3$. Suppose that $\mathbf{u}$ is prescribed on both the inner and outer boundary, viz., $\mathbf{u}(\mathbf{x}) = \alpha\mathbf{x}$ when $|\mathbf{x}| = a$ and $\mathbf{u}(\mathbf{x}) = \beta\mathbf{x}$ when $|\mathbf{x}| = b$, where $0 < \alpha a < \beta b$. Then for a large class of homogeneous, isotropic, stored-energy functions it has been shown (see, e.g., [4,51,56,57]) that there is a unique minimizer of the energy, $\mathbf{x} \mapsto \mathbf{u}_{\mathrm{m_r}}(\mathbf{x},\alpha,\beta)$, among all *radial deformations*, i.e., deformations that satisfy

$$\mathbf{u}_{\mathrm{r}}(\mathbf{x},\alpha,\beta) = \frac{r(R,\alpha,\beta)}{R}\mathbf{x}, \quad R := |\mathbf{x}| \tag{6.1}$$

for some absolutely continuous, strictly increasing function $R \mapsto r(R,\alpha,\beta)$. Moreover, it turns out that $\mathbf{u}_{\mathrm{m_r}}$ is a classical solution of the equilibrium equations. In addition, the continuous dependence of a solution (and its derivative) of a second-order ordinary differential equation on its initial data can be used to show that

$$\mathbf{u}_{\mathrm{m_r}}(\alpha,\beta,\mathbf{x}) \to \alpha\mathbf{x} \quad \text{uniformly in } C^1(\overline{\mathscr{B}};\mathbb{R}^2) \text{ as } \beta \to \alpha.$$

Thus, such solutions satisfy (5.1) with $\mathbf{F}_{\mathrm{o}} = \alpha\mathbf{I}$ and hence Theorem 5.1 implies that when the radial minimizer of the energy, $\mathbf{u}_{\mathrm{m_r}}$, is sufficiently close (in C^1) to $\alpha\mathbf{x}$, it is the unique absolute minimizer of the energy among all admissible deformations, not just the radial ones.

Remark 6.1 For compressible neo-Hookean materials (see (6.2)) in 2-dimensions, more general results have been obtained in [32] and [53]. Both papers show that the radial minimizer, $\mathbf{u}_{\mathrm{m_r}}$, is the unique absolute minimizer of the energy among admissible deformations, whether or not $\mathbf{u}_{\mathrm{r}}$ is close to a homogeneous deformation. See, e.g., [10,33,34,44,47] for additional nonradial, equilibrium solutions to this problem. Each of the nonradial solutions in these papers generates a distinct mirror image solution with the same energy. Therefore, Theorem 4.2 implies that the nonradial solutions must violate (4.6).

6.2 Uniqueness for a Rectangle in Uniaxial Tension

Let $n = 2$. In this subsection we restrict our attention to *compressible neo-Hookean materials*[17], i.e., constitutive relations of the form:

$$W(\mathbf{F}) = \frac{\omega_{\mathrm{o}}}{2}|\mathbf{F}|^2 + h(\det\mathbf{F}), \qquad \mathbf{S}(\mathbf{F}) = \omega_{\mathrm{o}}\mathbf{F} + h'(\det\mathbf{F})\operatorname{cof}\mathbf{F}, \tag{6.2}$$

where $\omega_{\mathrm{o}} > 0$ denotes a constant and $h \in C^2(\mathbb{R}^{\succ};\mathbb{R}^{\succeq})$ satisfies

$$h'(1) = -\omega_{\mathrm{o}}, \qquad h''(1) > 0, \qquad h'' \geq 0, \qquad \lim_{t\to 0^+} h(t) = +\infty. \tag{6.3}$$

We will use Proposition 4.7 to construct two examples, one for a mixed problem and one for a traction problem, of homogeneous equilibrium solutions that are each the unique

[17] Blatz and Ko [11] showed that the experimental data of Bridgman [12], for certain solid rubbers, is compatible with $(6.2)_1$ with $h(t) = t^{-13.3}$; see p. 238 and equation (50) (with $f = 1$) in [11].

absolute minimizer of the energy when the stored-energy density is given by (6.2)–(6.3). In both examples the deformation will be of the form, for some $\alpha > 0$ and $\lambda > 0$,

$$\mathbf{u}_\mathrm{e}(x,y) := \mathbf{F}_\mathrm{e}\begin{bmatrix} x \\ y \end{bmatrix}, \qquad \mathbf{F}_\mathrm{e} := \begin{bmatrix} \alpha & 0 \\ 0 & \lambda \end{bmatrix}, \tag{6.4}$$

which is a classical solution of the equilibrium equations. If $\mathbf{u}_\mathrm{e}$ also satisfies appropriate boundary conditions, then **Proposition 4.7 implies that $\mathbf{u}_\mathrm{e}$ is the unique absolute minimizer of the energy whenever**

$$|h'(\alpha\lambda)| < \omega_\mathrm{o}. \tag{6.5}$$

Let the body, in its reference configuration, occupy the rectangle $\overline{\mathscr{B}} := [-X,X] \times [-Y,Y]$ for some $X > 0$ and $Y > 0$. Fix $\lambda > 1$ and consider the functional

$$\mathscr{E}(\alpha) = \frac{\omega_\mathrm{o}}{2}(\alpha^2 + \lambda^2) + h(\alpha\lambda),$$

which, up to a multiplicative constant, is the total energy of the body when it undergoes the deformation (6.4). Our hypotheses on h imply that $\mathscr{E} \in C^2(\mathbb{R}^{\succ};\mathbb{R}^{\succ})$ blows up at 0 and $+\infty$ and hence that $\mathscr{E}$ achieves its infimum at one or more $\alpha > 0$. Since

$$\mathscr{E}'(\alpha) = \omega_\mathrm{o}\alpha + \lambda h'(\alpha\lambda) = 0, \qquad \mathscr{E}''(\alpha) = \omega_\mathrm{o} + \lambda^2 h''(\alpha\lambda) > 0, \tag{6.6}$$

we conclude from the strict convexity of $\mathscr{E}$ that $\alpha = \alpha(\lambda)$ is unique and satisfies $(6.6)_1$ (which is the condition that the sides of the rectangle, $x = \pm X$, are free of tractions). For future reference we note that that $(6.6)_1$ implies that $h'(\alpha(\lambda)\lambda) = -\omega_\mathrm{o}\alpha(\lambda)/\lambda$ and **hence that (6.5) will follow whenever**

$$\alpha(\lambda) < \lambda. \tag{6.7}$$

We now establish the following result.

Proposition 6.2 *Fix $\lambda > 1$, $\mathscr{B} := (-X,X) \times (-Y,Y) \subset \mathbb{R}^2$, and*

$$E(\mathbf{u}) := \int_{-Y}^{Y}\int_{-X}^{X} W\big(\nabla\mathbf{u}(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y,$$

where W is given by (6.2)–(6.3)*. Then the unique absolute minimizer of E among deformations $\mathbf{u} \in \mathscr{A}$ with*

$$\mathbf{d}(x,-Y) := \begin{bmatrix} \alpha x \\ -\lambda Y \end{bmatrix}, \qquad \mathbf{d}(x,Y) := \begin{bmatrix} \alpha x \\ \lambda Y \end{bmatrix}, \quad -X < x < X, \tag{6.8}$$

is the homogeneous deformation $\mathbf{u}_\mathrm{e}$ given by (6.4) *with $\alpha = \alpha(\lambda)$ given by* $(6.6)_1$*. Moreover, there are no other weak equilibrium solutions $\mathbf{v}_\mathrm{e} = \mathbf{v}_\mathrm{e}(x,y)$ that satisfy*

$$\big|h'\big(\det\nabla\mathbf{v}_\mathrm{e}(x,y)\big)\big| < \omega_\mathrm{o} \;\; \textit{for a.e. } (x,y) \in \mathscr{B}.$$

Remark 6.3 Assumptions (6.2)–(6.3) are sufficient to apply the existence theory of Ball and Murat [7]. Thus, their minimizer is $\mathbf{u}_\mathrm{e}$.

Proof of Proposition 6.2 It is clear that $\mathbf{u}_e \in \mathscr{A}$ is a classical equilibrium solution that satisfies the displacement boundary condition $\mathbf{u}_e = \mathbf{d}$, where $\mathbf{d}$ is given by (6.8). Thus, as we previously noted, all that we need to show is that (6.7) is satisfied.

If we substitute $\alpha = \alpha(\lambda)$ into $(6.6)_1$ and then differentiate with respect to λ we find that

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\lambda} = -\frac{h'(\alpha\lambda) + \alpha\lambda h''(\alpha\lambda)}{\omega_o + \lambda^2 h''(\alpha\lambda)}, \tag{6.9}$$

where $\alpha = \alpha(\lambda)$. Consequently,

$$\theta(\lambda) := \lambda^2 \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{\alpha}{\lambda}\right] = -\frac{2\alpha\lambda h''(\alpha\lambda)}{\omega_o + \lambda^2 h''(\alpha\lambda)} \leq 0, \tag{6.10}$$

where we have made use of (6.9), $(6.6)_1$, and $(6.3)_3$. In addition, when $\lambda = 1$ it follows that $\alpha = 1$; thus, $(6.3)_2$ and (6.10) yield $\theta(1) < 0$, which together with (6.10) implies (6.7). □

Remark 6.4 If $\lambda \in (0,1)$, then a unique $\alpha = \alpha(\lambda)$ that satisfies $(6.6)_1$ still exists, however, in this case $\lambda < \alpha(\lambda)$ and hence $|h'(\alpha\lambda)| > \omega_o$. Thus, for $\lambda \in (0,1)$, our results have no implications concerning either the uniqueness of $\mathbf{u}_e$ or whether or not $\mathbf{u}_e$ is an absolute minimizer of E. One might expect[18] that for X small (or, equivalently, Y large) there would be two buckled equilibrium solutions, each with lower energy than $\mathbf{u}_e$. Although one expects these solutions to both be absolute minimizers of E, this has yet to be proven.

Next, instead of the displacement boundary condition on the top and bottom of the rectangle, $y = \pm Y$, we apply tractions. For the deformation $\mathbf{u}_e$ (with $\alpha = \alpha(\lambda)$ given by $(6.6)_1$) we will specify

$$\mathscr{F}(\lambda) = \omega_o\lambda + \alpha(\lambda) h'\big(\lambda\alpha(\lambda)\big) \tag{6.11}$$

(the normal force per unit length applied at $y = \pm Y$). We first need to show that for each $\mathscr{F} > 0$ there is a unique $\lambda = \lambda(\mathscr{F}) > 1$ that satisfies (6.11). We start by using $(6.6)_1$ to eliminate $h'(\alpha\lambda)$ from (6.11); thus,

$$\mathscr{F}(\lambda) = \omega_o\lambda\Big(1 - \left[\frac{\alpha}{\lambda}\right]^2\Big) \text{ with } \alpha = \alpha(\lambda). \tag{6.12}$$

Next, in view of (6.10), $(6.3)_2$, and the continuity of h'', $\alpha(\lambda)/\lambda$ is strictly decreasing in a neighborhood of $\lambda = 1$. Consequently, for every $\lambda_o > 1$ and sufficiently small,

$$\frac{\alpha(\lambda)}{\lambda} \leq \frac{\alpha(\lambda_o)}{\lambda_o} < \frac{\alpha(1)}{1} = 1 \text{ for all } \lambda \geq \lambda_o, \tag{6.13}$$

which together with (6.12) implies that $\mathscr{F}(\lambda) \to +\infty$ as $\lambda \to +\infty$. We next differentiate (6.12) with respect to λ to conclude, with the aid of (6.10) and (6.13), that

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}\lambda} = \omega_o\Big(1 - \left[\frac{\alpha}{\lambda}\right]^2\Big) - 2\omega_o\lambda\frac{\alpha}{\lambda}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{\alpha}{\lambda}\right] > 0$$

for all $\lambda > 1$. Thus, we can consider λ as a function of $\mathscr{F}$. Since $\mathscr{F}(1) = \omega_o + h'(1) = 0$ it follows that for each $\mathscr{F} > 0$ there exists a unique $\lambda = \lambda(\mathscr{F}) > 1$ that satisfies (6.11). Equation (6.7) and the proof of Proposition 6.2 then yield:

[18] A slight change in the boundary conditions at $y = \pm L$ allows one to prove that buckled solutions exist. See [50] and also [39, Chapter 10].

Proposition 6.5 *Let* $\mathscr{F} > 0$, $\mathscr{B} = (-X,X) \times (-Y,Y) \subset \mathbb{R}^2$, *and*

$$E(\mathbf{u}) := \int_{-Y}^{Y} \int_{-X}^{X} W\big(\nabla \mathbf{u}(x,y)\big)\, \mathrm{d}x\, \mathrm{d}y - \mathscr{F} \int_{-X}^{X} \big[u_2(x,Y) + u_2(x,-Y)\big]\, \mathrm{d}x,$$

where W *is given by* (6.2)–(6.3)*. Then the unique absolute minimizer of* E *among deformations* $\mathbf{u} \in \mathscr{A}$ *that satisfy*

$$\int_{-Y}^{Y} \int_{-X}^{X} \mathbf{u}(x,y)\, \mathrm{d}x\, \mathrm{d}y = \mathbf{0}$$

is the homogeneous deformation $\mathbf{u}_{\mathrm{e}}$ *given by* (6.4)*, where* $\alpha = \alpha(\lambda)$ *is given by* $(6.6)_1$ *and* $\lambda = \lambda(\mathscr{F})$ *is the unique solution of* (6.11)*. Moreover, there are no other weak equilibrium solutions* $\mathbf{v}_{\mathrm{e}} = \mathbf{v}_{\mathrm{e}}(x,y)$ *that satisfy*

$$\big| h'\big(\det \nabla \mathbf{v}_{\mathrm{e}}(x,y)\big) \big| < \omega_{\mathrm{o}} \ \textit{for a.e.}\ (x,y) \in \mathscr{B}.$$

6.3 A Heterogenous Annulus

6.3.1 The Constitutive Relation; the Boundary-Value Problem; Existence of Minimizers

Let $n = 2$. We herein restrict our attention to a *generalization of the compressible neo-Hookean materials* considered in the previous subsection, viz., constitutive relations of the form:

$$W(\mathbf{x},\mathbf{F}) = \frac{\omega(\mathbf{x})}{2} |\mathbf{F}|^2 + h(\mathbf{x}, \det \mathbf{F}), \qquad \mathbf{S}(\mathbf{x},\mathbf{F}) = \omega(\mathbf{x})\mathbf{F} + h'(\mathbf{x}, \det \mathbf{F}) \operatorname{cof} \mathbf{F}, \tag{6.14}$$

where, for some constant $\omega_{\mathrm{o}} > 0$ and every $t > 0$,

$$\omega \geq \omega_{\mathrm{o}} > 0 \ \textit{a.e.}, \qquad \omega \in L^\infty(\mathscr{B}), \qquad \mathbf{x} \mapsto h(\mathbf{x},t) \text{ is measurable}, \tag{6.15}$$

and for every $\tau \in (0,1)$ there exists a constant $K_\tau > 0$ such that, for every $t \in [\tau, 1/\tau]$,

$$h(\mathbf{x},t) \leq K_\tau \ \text{ for } \textit{a.e.}\ \mathbf{x} \in \mathscr{B}. \tag{6.16}$$

In addition we assume that, for almost every $\mathbf{x} \in \mathscr{B}$,

$$\begin{gathered} t \mapsto h(\mathbf{x},t) \in C^2(\mathbb{R}^{\succ}; \mathbb{R}^{\succeq}), \qquad h''(\mathbf{x},t) \geq 0, \qquad \lim_{t \to 0^+} h(\mathbf{x},t) = +\infty, \\ h'(\mathbf{x}, \det \mathbf{F}) := \frac{\partial}{\partial \lambda} h(\mathbf{x},\lambda)\Big|_{\lambda = \det \mathbf{F}}, \qquad h''(\mathbf{x}, \det \mathbf{F}) := \frac{\partial^2}{\partial \lambda^2} h(\mathbf{x},\lambda)\Big|_{\lambda = \det \mathbf{F}}. \end{gathered} \tag{6.17}$$

We now fix $b > a > 0$ and $\alpha, \beta \in \mathbb{R}^{\succ}$, with $\beta b > \alpha a$, and consider the energy

$$E(\mathbf{u}) := \int_A W\big(\mathbf{x}, \nabla \mathbf{u}(\mathbf{x})\big)\, \mathrm{d}\mathbf{x}, \qquad A := \{\mathbf{x} \in \mathbb{R}^2 : a < |\mathbf{x}| < b\}, \tag{6.18}$$

where W is given by $(6.14)_1$ and satisfies (6.15)–(6.17). We consider minimizing E among admissible deformations $\mathbf{u} \in \mathscr{A}$ that satisfy the boundary conditions

$$\mathbf{u}(\mathbf{x}) = \alpha \mathbf{x} \ \text{ for } \ |\mathbf{x}| = a, \qquad \mathbf{u}(\mathbf{x}) = \beta \mathbf{x} \ \text{ for } \ |\mathbf{x}| = b. \tag{6.19}$$

A result of Müller [40] or Zhang [62] immediately yields:

Proposition 6.6 *Let $\mathscr{B} = A$. Then there exists a $\mathbf{u}_{\mathrm{m}} \in \mathscr{A}$ that is an absolute minimizer of E among those admissible deformations that satisfy* (6.19)*. Furthermore, there also exists an admissible radial deformation $\mathbf{u}_{\mathrm{m_r}} \in \mathscr{A}$ (see* (6.1)*) that is an absolute minimizer of the energy among radial deformations that satisfy* (6.19)*.*

Proof Hypotheses (6.14), (6.15), and (6.17) imply that all except one of the required conditions of [40, Theorem 5.1] and [62, Theorem 3.1] are satisfied. Thus, the existence of an absolute minimizer follows, provided we exhibit an admissible deformation with finite energy. Here we make use of (6.16). An admissible deformation with finite energy is given by (6.1) with

$$r(R,\alpha,\beta) := \left[\frac{b-R}{b-a}\right]\alpha a + \left[\frac{R-a}{b-a}\right]\beta b = \left[\frac{\beta b - \alpha a}{b-a}\right]R - \frac{(\beta-\alpha)ab}{b-a}, \quad a \le R \le b.$$

Finally, the last statement of the proposition follows from the fact that the set of radial deformations is a weakly closed subset of $\mathscr{A}$ (since it is a convex, closed subset of $\mathscr{A}$). □

6.3.2 A Simple Composite; Uniqueness of the Absolute Minimizer

In this subsection we take $n = 2$ and restrict our attention to a simple, two-phase body. Each phase is assumed to be composed of a compressible neo-Hookean material. We take $\mathscr{B} = A$, the annulus of the previous subsection. Let $c \in (a,b)$ and suppose that its inner portion, $a \le |\mathbf{x}| \le c$ (the *minus* side), is composed of one such material, while the outer portion, $c \le |\mathbf{x}| \le b$ (the *plus* side), is composed of a second such material; thus

$$W_\pm(\mathbf{F}) = \frac{\omega_\pm}{2}|\mathbf{F}|^2 + h_\pm(\det\mathbf{F}), \qquad \mathbf{S}_\pm(\mathbf{F}) = \omega_\pm\mathbf{F} + h'_\pm(\det\mathbf{F})\,\mathrm{cof}\,\mathbf{F}, \tag{6.20}$$

where $\omega_- > 0$ and $\omega_+ > 0$ denote constants and $h_-, h_+ \in C^2(\mathbb{R}^\succ;\mathbb{R}^\succeq)$ satisfy

$$h'_\pm(1) = -\omega_\pm, \qquad h''_\pm > 0, \qquad \lim_{t\to 0^+} h_\pm(t) = +\infty. \tag{6.21}$$

The annulus can be *viewed as a single generalized neo-Hookean body* (see the previous subsection) with

$$W(\mathbf{x},\mathbf{F}) := W_-(\mathbf{F}) \ \text{ if } |\mathbf{x}| < c, \qquad W(\mathbf{x},\mathbf{F}) := W_+(\mathbf{F}) \ \text{ if } |\mathbf{x}| > c. \tag{6.22}$$

Instead, we will outline an alternate approach to this problem. Let γ satisfy $\alpha a < \gamma c < \beta b$. Consider the energies

$$E_\pm(\mathbf{u}^\pm,\gamma) := \int_{A_\pm} W_\pm\big(\nabla\mathbf{u}^\pm(\mathbf{x})\big)\,\mathrm{d}\mathbf{x}, \qquad \begin{array}{l} A_- := \{\mathbf{x}\in A : a < |x| < c\}, \\ A_+ := \{\mathbf{x}\in A : c < |x| < b\} \end{array}$$

and suppose that the deformations are subject to the boundary conditions

$$\mathbf{u}^-(\mathbf{x}) = \alpha\mathbf{x} \ \text{ for } \ |\mathbf{x}| = a, \quad \mathbf{u}^+(\mathbf{x}) = \gamma\mathbf{x} \ \text{ for } \ |\mathbf{x}| = c, \quad \mathbf{u}^+(\mathbf{x}) = \beta\mathbf{x} \ \text{ for } \ |\mathbf{x}| = b. \tag{6.23}$$

Then, as in §6.1, there exist admissible radial deformations $\mathbf{u}^-_{\mathrm{m_r}}(\mathbf{x},\alpha,\gamma)$ and $\mathbf{u}^+_{\mathrm{m_r}}(\mathbf{x},\beta,\gamma)$ that minimize $E_\pm$ (among such deformations). *Moreover, $\mathbf{u}^-_{\mathrm{m_r}}$ and $\mathbf{u}^+_{\mathrm{m_r}}$ each satisfy the equilibrium equations and the mappings $(\mathbf{x},\alpha,\gamma) \mapsto \mathbf{u}^-_{\mathrm{m_r}}(\mathbf{x},\alpha,\gamma)$ and $(\mathbf{x},\beta,\gamma) \mapsto \mathbf{u}^+_{\mathrm{m_r}}(\mathbf{x},\beta,\gamma)$ are* *each contained in* C^2. Next, for fixed α and β, define a one-parameter family of admissible radial deformations of A:

$$\mathbf{u}_{\mathrm{r}}(\mathbf{x},\gamma) := \begin{cases} \mathbf{u}^-_{\mathrm{m_r}}(\mathbf{x},\alpha,\gamma) & \text{if } a \le |\mathbf{x}| \le c, \\ \mathbf{u}^+_{\mathrm{m_r}}(\mathbf{x},\beta,\gamma) & \text{if } c \le |\mathbf{x}| \le b. \end{cases}$$

It follows that, for each $\gamma \in (\alpha a/c, \beta b/c)$, $\mathbf{x} \mapsto \mathbf{u}_{\mathrm{r}}(\mathbf{x},\gamma)$ satisfies the equilibrium equations on $A_+ \cup A_-$ and is continuous across the circle $|\mathbf{x}| = c$. However, for arbitrary γ, the surface traction on the inner boundary of A_+ may not be equal to the surface traction on the outer boundary of A_-.

Finally, consider the functional $\mathscr{E} \in C^1((\alpha a/c, \beta b/c))$ defined by

$$\mathscr{E}(\gamma) := \int_{A_-} W_-\big(\nabla \mathbf{u}_{\mathrm{r}}(\mathbf{x},\gamma)\big)\,\mathrm{d}\mathbf{x} + \int_{A_+} W_+\big(\nabla \mathbf{u}_{\mathrm{r}}(\mathbf{x},\gamma)\big)\,\mathrm{d}\mathbf{x}. \tag{6.24}$$

In view of $(6.20)_1$, (6.24), and Jensen's inequality[19]

$$\frac{\mathscr{E}(\gamma)}{\mathbb{A}_\pm} \ge \fint_{A_\pm} h_\pm\big(\det \nabla \mathbf{u}_{\mathrm{r}}(\mathbf{x},\gamma)\big)\,\mathrm{d}\mathbf{x} \ge h_\pm\bigg(\fint_{A_\pm} \det \nabla \mathbf{u}_{\mathrm{r}}(\mathbf{x},\gamma)\,\mathrm{d}\mathbf{x}\bigg), \tag{6.25}$$

where $\mathbb{A}_\pm := \mathrm{area}(A_\pm)$. Note that the area of $\mathbf{u}_{\mathrm{r}}(A_-)$ ($\mathbf{u}_{\mathrm{r}}(A_+)$) goes to zero as $\gamma c \searrow \alpha a$ ($\gamma c \nearrow \beta b$). Consequently, by $(6.21)_3$, the right-hand side of (6.25) becomes infinite as γ approaches either endpoint of $(\alpha a/c, \beta b/c)$. Since $\mathscr{E}$ is continuous it must therefore have an absolute minimizer $\gamma_{\mathrm{m}} = \gamma_{\mathrm{m}}(\alpha,\beta) \in (\alpha a/c, \beta b/c)$. Moreover, since $(\mathbf{x},\gamma) \mapsto \mathbf{u}^-_{\mathrm{m_r}}(\mathbf{x},\alpha,\gamma)$ and $(\mathbf{x},\gamma) \mapsto \mathbf{u}^+_{\mathrm{m_r}}(\mathbf{x},\beta,\gamma)$ are each C^2 (for each fixed α and β) and radial we can take the derivative[20] of $\mathscr{E}$ to conclude that

$$\begin{aligned} 0 = \frac{\mathrm{d}\mathscr{E}}{\mathrm{d}\gamma}(\gamma_{\mathrm{m}}) = \int_{A_-} &\mathbf{S}_-\big(\nabla \mathbf{u}_{\mathrm{r}}(\mathbf{x},\gamma_{\mathrm{m}})\big) : \nabla \mathbf{u}'_{\mathrm{r}}(\mathbf{x},\gamma_{\mathrm{m}})\,\mathrm{d}\mathbf{x} \\ &+ \int_{A_+} \mathbf{S}_+\big(\nabla \mathbf{u}_{\mathrm{r}}(\mathbf{x},\gamma_{\mathrm{m}})\big) : \nabla \mathbf{u}'_{\mathrm{r}}(\mathbf{x},\gamma_{\mathrm{m}})\,\mathrm{d}\mathbf{x}, \end{aligned} \tag{6.26}$$

where the prime here denotes the derivative with respect to γ. In order to simplify the right-hand side of (6.26), we next differentiate (6.23) with respect to γ to arrive at

$$\mathbf{u}'_{\mathrm{r}}(\mathbf{x},\gamma) \equiv \mathbf{0} \ \text{ for } \ |\mathbf{x}| = a, \quad \mathbf{u}'_{\mathrm{r}}(\mathbf{x},\gamma) = \mathbf{x} \ \text{ for } \ |\mathbf{x}| = c, \quad \mathbf{u}'_{\mathrm{r}}(\mathbf{x},\gamma) \equiv \mathbf{0} \ \text{ for } \ |\mathbf{x}| = b. \tag{6.27}$$

We now observe that $\mathbf{u}^\pm_{\mathrm{m_r}}$ each satisfy the equilibrium equations, i.e., $\mathrm{Div}\,\mathbf{S}_\pm = \mathbf{0}$. This together with (6.27), the identity $\mathrm{Div}\,\mathbf{S}^{\mathrm{T}}\mathbf{u} = \mathbf{S} : \nabla \mathbf{u} + \mathbf{u} \cdot \mathrm{Div}\,\mathbf{S}$, and the divergence theorem shows that (6.26) implies (cf. (2.3))

$$0 = \int_{|\mathbf{x}|=c} \mathbf{S}_-\big(\nabla \mathbf{u}_{\mathrm{r}}(\mathbf{x},\gamma_{\mathrm{m}})\big)\mathbf{n}^- \cdot \mathbf{x}\,\mathrm{d}\mathscr{H}^1_{\mathbf{x}} + \int_{|\mathbf{x}|=c} \mathbf{S}_+\big(\nabla \mathbf{u}_{\mathrm{r}}(\mathbf{x},\gamma_{\mathrm{m}})\big)\mathbf{n}^+ \cdot \mathbf{x}\,\mathrm{d}\mathscr{H}^1_{\mathbf{x}}, \tag{6.28}$$

where $\mathbf{n}^\pm$ denotes the outward unit normal to $A_\pm$. However, $\mathbf{u}_{\mathrm{r}}$ is radial and $\mathbf{n}^+ = -\mathbf{n}^-$; consequently, (6.28) reduces to

$$\mathbf{S}_-\big(\nabla \mathbf{u}_{\mathrm{r}}(\mathbf{x},\gamma_{\mathrm{m}})\big)\mathbf{n}^- = \mathbf{S}_+\big(\nabla \mathbf{u}_{\mathrm{r}}(\mathbf{x},\gamma_{\mathrm{m}})\big)\mathbf{n}^-$$

[19] The symbol $\fint$ here denotes the integral of the indicated function divided by the area of the region of integration.

[20] At an arbitrary minimizer this is not possible due to the Jacobian becoming negative with an additive variation. However, at a radial deformation this difficulty can be overcome. See, e.g., [4, §7.3], [51, pp. 133-135], or [56, Theorem 2.6.19] for details.

and hence $\mathbf{u}_{\mathrm{r}}(\cdot,\gamma_{\mathrm{m}})$ satisfies the equilibrium equations on all of A. We can now apply Theorem 4.2 whenever $\mathbf{u}_{\mathrm{r}}(\cdot,\gamma_{\mathrm{m}})$ satisfies both of the inequalities:

$$\begin{aligned} &\left|h'_-\big(\det\nabla\mathbf{u}_{\mathrm{r}}(\mathbf{x},\gamma_{\mathrm{m}})\big)-\nu\right|<\omega_- \ \text{ for every } \mathbf{x}\in\overline{A}_-,\\ &\left|h'_+\big(\det\nabla\mathbf{u}_{\mathrm{r}}(\mathbf{x},\gamma_{\mathrm{m}})\big)-\nu\right|<\omega_+ \ \text{ for every } \mathbf{x}\in\overline{A}_+ \end{aligned} \tag{6.29}$$

for some constant $\nu\in\mathbb{R}$. Therefore, if $\mathbf{u}_{\mathrm{r}}(\cdot,\gamma_{\mathrm{m}})$ satisfies (6.29), then $\mathbf{u}_{\mathrm{r}}(\cdot,\gamma_{\mathrm{m}})$ is the unique absolute minimizer of the energy and hence $\mathbf{u}_{\mathrm{r}}(\cdot,\gamma_{\mathrm{m}})$ is the function whose existence was established in Proposition 6.6. Moreover, the absolute minimizer of the energy among all admissible deformations is the same mapping as the absolute minimizer among radial deformations. In particular, we have the following uniqueness result.

Proposition 6.7 *There exists an $\varepsilon>0$ such that for every $\alpha,\beta\in(1-\varepsilon,1+\varepsilon)$ the radial equilibrium solution $\mathbf{x}\mapsto\mathbf{u}_{\mathrm{r}}(\mathbf{x},\gamma_{\mathrm{m}}(\alpha,\beta))$ is the unique absolute minimizer of the energy* (6.18) *with W given by* (6.20)–(6.22).

Proof We will first show that $\mathbf{u}(\mathbf{x})\equiv\mathbf{x}$ satisfies (6.29) and, hence, that it is the unique absolute minimizer[21] among deformations that satisfy the boundary conditions (6.19) with $\alpha=\beta=1$. The uniform continuity of $(\mathbf{x},\alpha,\gamma)\mapsto\nabla\mathbf{u}^-_{\mathrm{m_r}}(\mathbf{x},\alpha,\gamma)$ and $(\mathbf{x},\beta,\gamma)\mapsto\nabla\mathbf{u}^+_{\mathrm{m_r}}(\mathbf{x},\beta,\gamma)$ then will yield a neighborhood, $\mathscr{N}$, of $(1,1,1)$ ($\alpha=1$, $\beta=1$, and $\gamma=1$) such that $(\alpha,\beta,\gamma)\in\mathscr{N}$ implies that $\mathbf{u}^-_{\mathrm{m_r}}(\cdot,\alpha,\gamma)$ and $\mathbf{u}^+_{\mathrm{m_r}}(\cdot,\beta,\gamma)$ satisfy (6.29). Finally, we will prove that $\gamma_{\mathrm{m}}=\gamma_{\mathrm{m}}(\alpha,\beta)$ is continuous at $\alpha=\beta=1$ and hence that $(\alpha,\beta,\gamma_{\mathrm{m}}(\beta,\alpha))\in\mathscr{N}$, when α and β are sufficiently close to 1.

Consider $\alpha=\beta=1$. In view of $(6.20)_2$ and $(6.21)_1$, $\mathbf{S}_\pm(\mathbf{I})=\mathbf{0}$. Thus, $\mathbf{u}_{\mathrm{r}}(\mathbf{x},1)\equiv\mathbf{x}$ is an equilibrium solution that satisfies (6.19). Define $\nu:=-\min\{\omega^+,\omega^-\}$. Then, by $(6.21)_1$,

$$\begin{aligned} &|h'_+(1)-\nu|=\omega^+-\min\{\omega^+,\omega^-\}\le\omega^+-\omega^-<\omega^+,\\ &|h'_-(1)-\nu|=\omega^--\min\{\omega^+,\omega^-\}\le\omega^--\omega^+<\omega^- \end{aligned}$$

and hence $\mathbf{u}_{\mathrm{r}}(\mathbf{x},1)\equiv\mathbf{x}$ satisfies (6.29). Consequently, by Theorem 4.2, $\mathbf{u}_{\mathrm{r}}(\mathbf{x},\gamma_{\mathrm{m}})\equiv\mathbf{x}$ with $\gamma_{\mathrm{m}}=1$ is the unique absolute minimizer of E. Therefore, as previously stated, the uniform continuity of $\nabla\mathbf{u}^\pm_{\mathrm{m_r}}$ yields a neighborhood, $\mathscr{N}$, of $(1,1,1)$ such that $(\alpha,\beta,\gamma)\in\mathscr{N}$ implies that $\mathbf{u}^\pm_{\mathrm{m_r}}$ satisfy (6.29).

Finally, we claim that $\gamma_{\mathrm{m}}=\gamma_{\mathrm{m}}(\alpha,\beta)$ is continuous at $\alpha=\beta=1$. Suppose not. Then there exist sequences $\alpha_i\to1$ and $\beta_i\to1$ such that $\gamma_i:=\gamma_{\mathrm{m}}(\alpha_i,\beta_i)\nrightarrow\gamma_{\mathrm{m}}(1,1)=1$. Since $\gamma_i\in(\alpha_i a/c,\beta_i b/c)$ is bounded, there is a subsequence (not relabeled) $\gamma_i\to\gamma_{\mathrm{o}}$ for some $\gamma_{\mathrm{o}}\neq1$. However, the functional $\mathscr{E}$ defined in (6.24) is a continuous function of α, β, and γ. More precisely,

$$\mathscr{E}(\alpha,\beta,\gamma):=\int_{A_-}W_-\big(\nabla\mathbf{u}^-_{\mathrm{m_r}}(\mathbf{x},\alpha,\gamma)\big)\,\mathrm{d}\mathbf{x}+\int_{A_+}W_+\big(\nabla\mathbf{u}^+_{\mathrm{m_r}}(\mathbf{x},\beta,\gamma)\big)\,\mathrm{d}\mathbf{x}$$

is continuous and hence $\mathscr{E}(\alpha_i,\beta_i,\gamma_i)\to\mathscr{E}(1,1,\gamma_{\mathrm{o}})$. However, $\mathbf{u}_{\mathrm{r}}(\mathbf{x},1)\equiv\mathbf{x}$ is the unique absolute minimizer of the energy when $\alpha=\beta=1$: $\mathscr{E}(1,1,\gamma_{\mathrm{o}})>\mathscr{E}(1,1,1)$. Define

$$\eta:=\big[\mathscr{E}(1,1,\gamma_{\mathrm{o}})-\mathscr{E}(1,1,1)\big]/2.$$

Then, in view of the continuity of $\mathscr{E}$, it follows that, for i sufficiently large,

$$\big|\mathscr{E}(\alpha_i,\beta_i,\gamma_i)-\mathscr{E}(1,1,\gamma_{\mathrm{o}})\big|<\eta,\qquad\big|\mathscr{E}(\alpha_i,\beta_i,1)-\mathscr{E}(1,1,1)\big|<\eta.$$

Consequently, $\mathscr{E}(\alpha_i,\beta_i,1)<\mathscr{E}(\alpha_i,\beta_i,\gamma_i)$, which contradicts the definition of $\gamma_i:=\gamma_{\mathrm{m}}(\alpha_i,\beta_i)$. Thus, γ_{m} must be continuous at $(1,1)$. □

[21] Modulo a rotation, $\mathbf{I}$ is in fact the unique absolute minimizer of $\mathbf{F}\mapsto W_\pm(\mathbf{F})$.

Remark 6.8 Suppose that, in addition to (6.21), $h_\pm$ satisfy[22] $h'_\pm(t) < 0$ for all t. Suppose further that $\alpha \le 1$ and $\beta \ge 1$. Then the radial minimizer *might* be expected to satisfy (6.29) with $\nu = 0$ for all such α and β. If this is indeed true, then all such radial minimizers would be unique absolute minimizers.

Remark 6.9 An analysis comparable to the one presented here shows that an annulus composed of many annular layers of different neo-Hookean materials (or alternating layers of two such materials) also has a unique energy minimizer, which is radial, when (α,β) is close to $(1,1)$. This is clear if the reference configuration is stress-free and the body has layers with k distinct neo-Hookean materials; in this case

$$\nu := -\min\{\omega_1,\omega_2,\omega_3,\ldots,\omega_k\}.$$

If the reference configuration is at equilibrium, but not stress-free, i.e., $h_i'(1) \neq -\omega_i$ for one or more layers, then in order for the reference configuration to be at equilibrium one must have, for some constant C,

$$h_i'(1) + \omega_i = C \quad \text{for} \;\; i = 1,2,3,\ldots,k.$$

In this case the choice

$$\nu := C - \min\{\omega_1,\omega_2,\omega_3,\ldots,\omega_k\}$$

will yield a reference configuration that satisfies (6.29).

Remark 6.10 The constitutive relation that we have used in this section can clearly be generalized to any homogeneous, isotropic material for which a radial existence and regularity theory is valid. See, e.g., [4,51,56,57] for examples. A similar analysis will, most likely, remain valid for many anisotropic constitutive relations for which one can establish a radial existence and regularity theory. See, e.g., [2]. Analogous results are valid for 3-dimensional spherical shells. The main difficulty there is that the term $|\mathbf{F}|^2$ in the stored-energy density as well as condition (4.6) must then be replaced by $|\mathbf{F}|^p$ with $p \ge 3$ and the more complicated condition (4.7). (However, the choice $p = 3$ greatly simplifies (4.7). See (1.3).)

7 An Alternate Approach to Uniqueness

Recently, Gao, Neff, Roventa, and Thiel [24] have established an interesting alternative approach to proving uniqueness. Assuming the stored-energy density, W, is homogeneous and invariant under a change in observer, a standard result yields a function[23] $\widehat{W} \in C^1(\mathrm{Psym}_n)$ that satisfies $\widehat{W}(\mathbf{C}) = W(\mathbf{F})$, where $\mathbf{C} := \mathbf{F}^{\mathrm{T}}\mathbf{F}$. The *second Piola-Kirchhoff stress tensor* $\mathbf{K} : \mathrm{Psym}_n \to \mathrm{Sym}_n$ is defined by

$$\mathbf{K}(\mathbf{C}) := 2\frac{\mathrm{d}\widehat{W}}{\mathrm{d}\mathbf{C}}(\mathbf{C}) = \mathbf{F}^{-1}\mathbf{S}(\mathbf{F}).$$

The proof in [24, Proposition 2.1] implies[24] the following:

Proposition 7.1 *Let* $\mathbf{u}_\mathrm{e}$ *be a weak equilibrium solution. Define* $\mathbf{C}_\mathrm{e}(\mathbf{x}) := [\nabla\mathbf{u}_\mathrm{e}(\mathbf{x})]^{\mathrm{T}}[\nabla\mathbf{u}_\mathrm{e}(\mathbf{x})]$. *Suppose that, for a.e.* $\mathbf{x} \in \mathscr{B}$,

22 For example, $h'(t) = -\omega_o t^{-m}$, $m > 0$, cf. footnote 17.

23 Recall that Psym_n denotes the set of strictly positive-definite, symmetric $\mathbf{C} \in \mathrm{Lin}_n$.

24 The uniqueness of the absolute minimizer also requires a result of Ciarlet and Mardare [15, Theorem 2.1].

1. $\mathbf{C}_e(\mathbf{x})$ *is a point of convexity of* $\widehat{W}$ *(see footnote 12); and*
2. $\mathbf{K}(\mathbf{C}_e(\mathbf{x}))$ *is positive semi-definite.*

Then $\mathbf{u}_e$ *is an absolute minimizer of* E*. Suppose, in addition, that* $\mathscr{D} \neq \varnothing$*, for a.e.* $\mathbf{x} \in \mathscr{B}$*,* $\mathbf{C}_e(\mathbf{x})$ *is a point of strict convexity of* $\widehat{W}$*, and* $\mathbf{u}_e \in C^1(\mathscr{B};\mathbb{R}^n)$ *satisfies* $\det \nabla \mathbf{u}_e > 0$ *on* $\mathscr{B}$*, then* $\mathbf{u}_e$ *is the unique absolute minimizer of* E*.*

In particular, when $\widehat{W}$ is convex on its entire domain, Psym_n, then one only needs to consider the sign of the eigenvalues of the second Piola-Kirchhoff stress tensor $\mathbf{K}$. The main difficulty with Proposition 7.1 is the required convexity of the function $\mathbf{C} \mapsto \widehat{W}(\mathbf{C})$. Such an assumption is independent of the polyconvexity of W; it is neither necessary nor sufficient for the existence[25] of minimizers. A minor additional problem is the smoothness assumption on $\mathbf{u}_e$ that is required for uniqueness. However, when $\widehat{W}$ is in fact convex, Proposition 7.1 may yield results that are better than those produced by Theorem 4.2.

A An Optimal Constant

We have made use of a variant of a result of Evans [20, pp. 248–250].

Proposition A.1 *Let* $p \in [2,\infty)$*. Then there exists a constant* $\kappa_p \in [2^{2-p}, p2^{1-p}]$ *such that*

$$|\mathbf{a}|^p \geq |\mathbf{b}|^p + p|\mathbf{b}|^{p-2}\mathbf{b}\cdot(\mathbf{a}-\mathbf{b}) + \sigma\kappa_p|\mathbf{a}-\mathbf{b}|^p + (1-\sigma)\frac{p}{2}|\mathbf{b}|^{p-2}|\mathbf{a}-\mathbf{b}|^2 \tag{A.1}$$

for all $\mathbf{a},\mathbf{b} \in \mathbb{R}^n$ *and* $\sigma \in [0,1]$*.*

Remark A.2 If $p = 2$ then (A.1) reduces to $|\mathbf{a}|^2 = |\mathbf{b}|^2 + 2\mathbf{b}\cdot(\mathbf{a}-\mathbf{b}) + |\mathbf{a}-\mathbf{b}|^2$; thus $\kappa_2 = 1$. Also, it follows from [41] that $\kappa_3 = 2-\sqrt{2}$.

Proof of Proposition A.1 (cf. the proof of Proposition A.1 in [41]) In view of the previous remark we assume that $p > 2$. We note that when $\sigma = 1$ this result can be found in Müller et al. [41, Appendix]. We will prove the result when $\sigma = 0$. Inequality (A.1) will then follow upon taking a convex combination of the resulting inequalities.

Let $\sigma = 0$ and $p > 2$. If $\mathbf{b} = \mathbf{0}$ then (A.1) is clear. By homogeneity we may assume $|\mathbf{b}| = 1$. Therefore, (A.1) will follow if we show that, for all $\mathbf{a},\mathbf{b} \in \mathbb{R}^n$ with $|\mathbf{b}| = 1$,

$$|\mathbf{a}|^p \geq 1 + p\mathbf{b}\cdot(\mathbf{a}-\mathbf{b}) + \frac{p}{2}|\mathbf{a}-\mathbf{b}|^2. \tag{A.2}$$

Suppose now that $n = 1$ and define $t := \mathrm{sgn}(b)(a-b)$. Then (A.2) reduces to the inequality

$$\phi(t) := |t+1|^p - 1 - pt - \frac{p}{2}t^2 \geq 0. \tag{A.3}$$

If we differentiate ϕ we find that, for $t \neq -1$,

$$\phi'(t) = p(t+1)\big[\mathrm{sgn}(t+1)|t+1|^{p-2} - 1\big].$$

The only possible solutions of $\phi'(t) = 0$ are $t = -1$ and $t = 0$. Since $\phi(0) = 0$, $\phi(-1) = (p-2)/2 > 0$, and ϕ is a continuous function that blows up at $\pm\infty$ inequality (A.3) follows.

Next, let $\sigma = 0$, $p > 2$, $|\mathbf{b}| = 1$, and $n > 1$. We write $\mathbf{a} = \mathbf{b} + t\mathbf{e}$ and $\alpha = \mathbf{e}\cdot\mathbf{b}$, where $|\mathbf{e}| = 1$ and $\alpha \in [-1,1]$. Then (A.2) reduces to, for all $t \in \mathbb{R}$ and $\alpha \in [-1,1]$,

$$\theta(t,\alpha) := \big[1 + 2\alpha t + t^2\big]^{p/2} - 1 - p\alpha t - \frac{p}{2}t^2 \geq 0. \tag{A.4}$$

When $\alpha = \pm 1$ the vectors $\mathbf{a}$ and $\mathbf{b}$ are colinear; (A.4) then follows from the above argument with $n = 1$. For fixed t we differentiate θ with respect to α and set the result equal to zero to conclude that $\alpha = -t/2$. However, $\theta(t,-t/2) = 0$, which establishes (A.4) and completes the proof of (A.1). □

[25] However, see [16] where the implicit function theorem is used to obtain existence for small data.

B Determinants and Cofactors

B.1 The Determinant of a Sum

A standard identity[26] is for every $\mathbf{F},\mathbf{H}\in\mathrm{Lin}_n$,

$$\det(\mathbf{H}+\mathbf{F})=\begin{cases}\det\mathbf{F}+\mathbf{H}:\operatorname{cof}\mathbf{F}+\det\mathbf{H}, & \text{if } n=2,\\ \det\mathbf{F}+\mathbf{H}:\operatorname{cof}\mathbf{F}+\mathbf{F}:\operatorname{cof}\mathbf{H}+\det\mathbf{H}, & \text{if } n=3.\end{cases} \tag{B.1}$$

In particular, the choice $\mathbf{H}=\mathbf{G}-\mathbf{F}$ yields

$$\det\mathbf{G}=\begin{cases}\det\mathbf{F}+[\mathbf{G}-\mathbf{F}]:\operatorname{cof}\mathbf{F}+\det(\mathbf{G}-\mathbf{F}), & \text{if } n=2,\\ \det\mathbf{F}+[\mathbf{G}-\mathbf{F}]:\operatorname{cof}\mathbf{F}+\mathbf{F}:\operatorname{cof}[\mathbf{G}-\mathbf{F}]+\det(\mathbf{G}-\mathbf{F}), & \text{if } n=3.\end{cases} \tag{B.2}$$

B.2 The Derivative of the Cofactor in 3-Dimensions

Lemma B.1 *Let* $\mathbf{F},\mathbf{H}\in\mathrm{Lin}_3$. *Then*

$$\mathbb{K}(\mathbf{F})[\mathbf{H}]:=\frac{\mathrm{d}(\operatorname{cof}\mathbf{F})}{\mathrm{d}\mathbf{F}}[\mathbf{H}]=\operatorname{cof}(\mathbf{F}+\mathbf{H})-\operatorname{cof}\mathbf{F}-\operatorname{cof}\mathbf{H}. \tag{B.3}$$

Proof Fix $\mathbf{F},\mathbf{H}\in\mathrm{Lin}_3$. Then a simple computation shows that $\operatorname{cof}(\mathbf{F}+t\mathbf{H})$ is a quadratic polynomial in $t\in\mathbb{R}$, i.e., there exist $\mathbf{A},\mathbf{B},\mathbf{C}\in\mathrm{Lin}_3$ such that, for all $t\in\mathbb{R}$,

$$\operatorname{cof}(\mathbf{F}+t\mathbf{H})=\mathbf{A}+\mathbf{B}t+\mathbf{C}t^2. \tag{B.4}$$

At $t=0$ we get $\operatorname{cof}\mathbf{F}=\mathbf{A}$. If we divide (B.4) by t^2 and let $t\to\infty$ we find that $\operatorname{cof}\mathbf{H}=\mathbf{C}$. If we differentiate (B.4) with respect to t and then let $t=0$ we conclude that

$$\frac{\mathrm{d}(\operatorname{cof}\mathbf{F})}{\mathrm{d}\mathbf{F}}[\mathbf{H}]=\mathbf{B}.$$

The desired result follows from (B.4) at $t=1$ together with our formulae for $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. □

Remark B.2 It is clear from the definition that $\mathbb{K}(\mathbf{F}):\mathrm{Lin}_3\to\mathrm{Lin}_3$ is a linear map for every $\mathbf{F}\in\mathrm{Lin}_3$. If we interchange $\mathbf{F}$ and $\mathbf{H}$ in (B.3) we find that $\mathbb{K}(\mathbf{F})[\mathbf{H}]=\mathbb{K}(\mathbf{H})[\mathbf{F}]$; consequently, $\mathbf{F}\mapsto\mathbb{K}(\mathbf{F})[\mathbf{H}]:\mathrm{Lin}_3\to\mathrm{Lin}_3$ is also linear. Thus, $(\mathbf{F},\mathbf{H})\mapsto\mathbb{K}(\mathbf{F})[\mathbf{H}]$ is bilinear.

B.3 An Upper Bound on the Cofactor in 3-Dimensions

Lemma B.3 *Let* $\mathbf{X},\mathbf{H}\in\mathrm{Lin}_3$. *Then*

$$|\mathbf{X}:\operatorname{cof}\mathbf{H}|\le\gamma|\mathbf{H}|^2\le|\mathbf{X}||\mathbf{H}|^2, \tag{B.5}$$

where γ denotes the largest eigenvalue of $\sqrt{\mathbf{X}^{\mathrm{T}}\mathbf{X}}$.

Proof We first note that $\mathbf{V}:=\sqrt{\mathbf{X}^{\mathrm{T}}\mathbf{X}}$ is symmetric and positive semi-definite and hence, by the spectral theorem, has eigenvalues $\gamma\ge\beta\ge\alpha\ge0$. Thus,

$$|\mathbf{X}|^2=|\mathbf{V}|^2=\alpha^2+\beta^2+\gamma^2\ge\gamma^2,$$

which establishes the second inequality in (B.5).

[26] See, e.g., [14, p. 51]. Alternatively, one can derive (B.1) from the characteristic polynomial.

We now apply the polar decomposition theorem to conclude $\mathbf{H} = \mathbf{QU}$ and $\mathbf{X} = \mathbf{RV}$, where $\mathbf{U}$ and $\mathbf{V}$ are symmetric and positive semi-definite and $\mathbf{Q}$ and $\mathbf{R}$ are orthogonal. Next, by the spectral theorem, there exists an orthonormal basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ and real numbers $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ such that

$$\begin{aligned} \mathbf{H} = \mathbf{QU} &= \lambda_1 \mathbf{Qf}_1 \otimes \mathbf{f}_1 + \lambda_2 \mathbf{Qf}_2 \otimes \mathbf{f}_2 + \lambda_3 \mathbf{Qf}_3 \otimes \mathbf{f}_3, \\ \operatorname{cof} \mathbf{H} = \mathbf{Q} \operatorname{cof} \mathbf{U} &= \lambda_2 \lambda_3 \mathbf{Qf}_1 \otimes \mathbf{f}_1 + \lambda_1 \lambda_3 \mathbf{Qf}_2 \otimes \mathbf{f}_2 + \lambda_1 \lambda_2 \mathbf{Qf}_3 \otimes \mathbf{f}_3, \\ |\mathbf{H}|^2 = |\mathbf{QU}|^2 &= |\mathbf{U}|^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2. \end{aligned} \tag{B.6}$$

Consequently,

$$\begin{aligned} (\operatorname{cof} \mathbf{H}) : \mathbf{X} &= \lambda_2 \lambda_3 \mathbf{Qf}_1 \cdot \mathbf{RVf}_1 + \lambda_1 \lambda_3 \mathbf{Qf}_2 \cdot \mathbf{RVf}_2 + \lambda_1 \lambda_2 \mathbf{Qf}_3 \cdot \mathbf{RVf}_3 \\ &= \lambda_2 \lambda_3 \mathbf{e}_1 \cdot \mathbf{Vf}_1 + \lambda_1 \lambda_3 \mathbf{e}_2 \cdot \mathbf{Vf}_2 + \lambda_1 \lambda_2 \mathbf{e}_3 \cdot \mathbf{Vf}_3, \end{aligned} \tag{B.7}$$

where $\mathbf{e}_i := \mathbf{R}^{\mathrm{T}} \mathbf{Qf}_i$, $i = 1, 2, 3$, denote unit vectors. However,

$$|\mathbf{e}_i \cdot \mathbf{Vf}_i| \leq |\mathbf{e}_i| \, |\mathbf{Vf}_i| \leq \gamma, \tag{B.8}$$

where γ denotes the largest eigenvalue of $\mathbf{V} = \sqrt{\mathbf{X}^{\mathrm{T}} \mathbf{X}}$. The desired result, (B.5), now follows from (B.6)–(B.8), together with the inequality $2\lambda_i \lambda_j \leq \lambda_i^2 + \lambda_j^2$. □

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