Triangular Sets Seminar

Notes by J.H. Davenport — J.H.Davenport@bath.ac.uk

29 August 2013

Though these definitions are about individual cells, what one really wants is a decomposition of $\mathbf{R}^n$ such that *every* cell has the relevant property.

Definition 1 *A cell C of dimension d is* well-bordered *if the entire boundary of C is contained in the closure of the $d-1$-dimensional components of the boundary.*

Definition 2 *A cell C of dimension d is* boundary-coherent *if the boundary of C is the union of cells of dimension at most $d-1$.*

The typical counter-example to both of these is the "flagpole" construction.

JHD The $\vdash$ example is bad, but it is not "well-bordered" Is it a good idea to restrict ourselves to these?

ME Is this always possible — what about Nicolai's example [BGV10, BGV13]?

AL That shows that you can't get **strongly bordered**. There was a debate about Lazard's theorem [Laz10].

GKS What about the "sphere less North Pole", a 2-cell whose boundary is a 0-cell, which is boundary-coherent, but not well-bordered. The projection of the North Pole is inside the projection of the sphere, so it's not cylindrical. W have to add the South Pole, and this changes the topology.

Claim 1 (AL) *Cylindrical and boundary-coherent implies well-bordered.*

GKS Let C be an r-cell, with a boundary component D of dimension $\leq r-2$ not in the closue of any $r-1$-cell of the boundary of C.

bad Consider the first dimension-reducing projection of C. This doesn't work, see the "punctured sphere example".

GKS The image of C, $\pi(C)$ is of dimension r or $r-1$, while $\pi(D)$ has dimension at most $r-2$. Hence the two can't be equal, and by cylindricity must then be disjoint.

JHD Consider the sphere less the East Pole — it projects onto the disc less one point, which is not a cell.

GKS Gysen homomorphisms seem to be the key. This will need to be proved by induction on the dimension, he thinks.

Note that there are properties like well-oriented and well-based, whichare needed for CAD *construction* to work, as well as the above definitions which are needed for adjacency etc. to work.

Example 1 (Generalised $\vdash$) *Consider $f := x(y^2 - x)$. One CAD of $\mathbf{R}^2$ that is sign-invariant for f is $\{x < 0\}$, $\{x = 0\}$ and five cells above $x > 0$.*

Note:

- that McCallum's CADW algorithm [McC98] will insist on replacing f by $\{x, y^2 - x\}$ (by insisting on primitiveness as well as square-freeness);

  It might be thought that McCallum's theory of equational constraints [McC01] says that the *final* lift need only be with respect to the original polynomial(s), since we already have delineability: ME subsequently pointed out that the relevant theorem is only for irreducible polynomials (? is this necessary). However QEPCAD does the final lift with respect to the square-free (and primitive) basis.

- this CAD is not *order-invariant* since $(0, 0)$ has order 2, not 1.

So what might our goals be?

1. Use adjacency to produce a "minimal" CAD, which would be on the lines of combining cells which were in the same class (F-sign-invariant, F-order-invariant, T-truth-invariant) and did not destroy cylindricity (or possibly block-cylindricity).

   Example 2 *Note that this is a laudable goal, consider the case of aligned non-intersecting circles, where we get a totally spurious partition of $\mathbf{R}^1$ corresponding to the real root of the resultant where the two circles meet in the complex plane.*

2. But, as in the case of $\vdash$ or Example 1, doing this adjacency-merging might result in a CAD which was no longer well-bordered. Therefore *maybe* we need to look at "adjacency-merging respecting the well-bordered condition".

3. especially in this case, it is not clear that starting with a CAD and doing such merging where allowed will produce a *unique* minimal CAD as a result.

DJW's comments

[He was unable to be present]

> In terms of the 'goals' section — I agree with 2, I think keeping well-borderedness is important (as shown by the non-touching adjacent lines). With regards to 3, my gut feeling is that merging will *not* produce a unique minimal CAD, especially for CADs of high dimension. We have strict conditions and many choices will violate cylindricity or WB-ness - and more importantly this violation depends on the induced CADs and, therefore, the merging done in earlier steps. So it may be beneficial to prevent an earlier merging to allow a later merging, taking the earlier merging would result in a certain 'minimal' CAD, but taking the later merging would result in a different 'minimal' CAD. I can't think of any examples off the top of my head but will have a think.

JHD I don't think we'd really considered mictures of merging and lifting. It's certainly not clear.

References

[BGV10] S. Basu, A. Gabrielov, and N. Vorobjov. Semi-monotone sets. `http://arxiv.org/abs/1004.5047`, 2010.

[BGV13] S. Basu, A. Gabrielov, and N. Vorobjov. Semi-monotone sets. *J. Eur. Math. Soc.*, 15:635–657, 2013.

[Laz10] D. Lazard. CAD and Topology of Semi-Algebraic Sets. *Math. Comput. Sci.*, 4:93–112, 2010.

[McC98] S. McCallum. An Improved Projection Operation for Cylindrical Algebraic Decomposition. *Quantifier Elimination and Cylindrical Algebraic Decomposition*, pages 242–268, 1998.

[McC01] S. McCallum. On Propagation of Equational Constraints in CAD-Based Quantifier Elimination. In B. Mourrain, editor, *Proceedings ISSAC 2001*, pages 223–230, 2001.