Looking at a set of equations

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Abstract. This working paper* describes our experiences with using the Gröbner-basis method [Buchberger, 1985] to solve some related systems of polynomial equations. While we have not yet been able to solve the system that was our primary motivation, we feel that these experiences may prove useful to others investigating Buchberger's algorithm in this context, especially when, as is the case for the system under investigation, the equations are highly structured. We conclude with some examples of the polynomials that we factored in the course of this investigation.

* This is a 1989 reprint of a paper originally published in 1987. We have taken the opportunity to correct a few typographical errors.

Introduction

This investigation* began when Professor S. Arnborg asked us whether the following system of equations had finitely many or infinitely many roots. He added that he had been unable to resolve this problem, either computationally or theoretically.

$$\begin{gathered}
x_1 + x_2 + x_3 + x_4 + x_5 + x_6;\\
x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_6 + x_6x_1;\\
x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_6 + x_5x_6x_1 + x_6x_1x_2;\\
x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_6 + x_4x_5x_6x_1 + x_5x_6x_1x_2 + x_6x_1x_2x_3;\\
x_1x_2x_3x_4x_5 + x_2x_3x_4x_5x_6 + x_3x_4x_5x_6x_1 + x_4x_5x_6x_1x_2 + x_5x_6x_1x_2x_3 + x_6x_1x_2x_3x_4;\\
x_1x_2x_3x_4x_5x_6 - 1;
\end{gathered}$$

The first thing we decided to do was to look at a slightly simpler problem, which we were assured by Prof. Arnborg (several hours of time with MAPLE) had only a finite number of solutions, viz. the case of five variables.

$$\begin{gathered}
x_1 + x_2 + x_3 + x_4 + x_5;\\
x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1;\\
x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2;\\
x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_1 + x_4x_5x_1x_2 + x_5x_1x_2x_3;\\
x_1x_2x_3x_4x_5 - 1;
\end{gathered}$$

Parenthetically, we observe that the case of four variables, viz.

$$\begin{gathered}
x_1 + x_2 + x_3 + x_4;\\
x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1;\\
x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2;\\
x_1x_2x_3x_4 - 1;
\end{gathered}$$

is rather different — there are an infinite family of solutions, which are deduced in less than one second by the standard REDUCE Gröbner-basis package as being the family

$$\begin{gathered}
x_3^2x_4^6 - x_3^2x_4^2 - x_4^4 + 1;\\
x_3^3x_4^2 + x_3^2x_4^3 - x_3 - x_4;\\
x_2x_4^4 - x_2 + x_4^5 - x_4;\\
x_2x_3 - x_2x_4 + x_3^2x_4^4 + x_3x_4 - 2x_4^2;\\
x_2^2 + 2x_2x_4 + x_4^2;\\
x_1 + x_2 + x_3 + x_4;
\end{gathered}$$

which is the union of the two families

$$x_3x_4 - 1, x_2 + x_4, x_1 + x_3$$

and

$$x_3x_4 + 1, x_2 + x_4, x_1 + x_3.$$

Our investigation of this went through several phases, each of which is described in one of the following paragraphs with a cryptic phrase, which is its key in the tables that follow.

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* Most of this work was performed while the author enjoyed the hospitality of NADA, the Department of Computer Science at KTH Stockholm. The author is grateful to Prof. Arnborg for proposing the initial problem, and to Lars Langemyr for much help and many fruitful conversations. Since then, he has fruitful electronic discussions with M. Möller (FernUniversität Hagen) and H. Melenk (ZIB Berlin). Work on this problem is progressing, and this report describes the "current state" of the investigation.

Investigations in Five Variables

Original

This refers to the Gröbner-base code distributed with REDUCE 3.3 as a "contributed package", and attributed to R. Gebauer, A.C. Hearn and M. Möller (with contributions to the distributed polynomial code by R. Kredel) in its comments. We ran this, with all the default settings (as well as some of the options, which did not seem to make a great deal of difference) in a specially constructed REDUCE with $6\frac{1}{2}$ Megabytes of memory on a SUN 3/75 at NADA. This job did not terminate, but ran out of memory on the 210th. H-polynomial with 38 critical pairs left to consider, having just constructed an H-polynomial of total degree 26 requiring 3 seconds to compute (the high-water marks reached were 68 remaining pairs, total degree 33, which was reached at the same time, and 120 seconds, with the largest numbers seen having 19 digits). There was some hope that the process was "nearly finished", since the high-water marks seemed to have been safely passed, but it was quite clear that there was a great deal of work to be done before any solution appeared.

Monomial

We observed that the run described in the previous section had generated H-polynomials with very large repeated factors, e.g. the last H-polynomial (of degree 26) had a factor of of x_5^{10}, and furthermore was very similar to a previously computed polynomial which had had a higher power of x_5 dividing it. This led us to resurrect an old idea [Davenport, 1985] of suppressing multiple monomial factors when they are found in any H-polynomial. This means, of course, that we may not in fact be computing the Gröbner-basis of the given ideal (though in the present case it is easy to see that no element of the Gröbner-basis itself can contain a monomial factor, for then it would vanish when that monomial had the value zero, whereas the ideal clearly does not include such points), but rather that of an ideal somewhere between the original ideal and its radical. Nevertheless, the ideal we do compute will have the same zeroes as the original ideal, and for many purposes this is sufficient. This is a fairly easy task to perform, and the reduction in complexity it can bring is substantial.

In our case, the savings were indeed dramatic. By the time we get to polynomial 14 (including the original five) we can save one factor of x_5, This in turn enables the algorithm to rule out one critical pair, which the original case could not, and needed 1.8 seconds to prove that it reduced to zero. Polynomial 19 again gains a factor of x_5, as do the next three (as a consequence of this simplification). Polynomial 23 gains a factor of x_5^2, as do the next two. By polynomial 26, we see a saving of x_4x_5, which is repeated in the next polynomial, and then grows to $x_4x_5^3$. This difference continues growing for a while, until polynomial 41 is reached, when the modified code can deduce a polynomial free of x_3, while the original one has to perform a further reduction before it can do so. The next time they are in step with only x_4 and x_5 to eliminate, the modified code is four steps ahead, as well as being a factor of $x_4^2x_5^5$ ahead, and having eight fewer critical pairs left to do. From this point on, the modified code steadily increases its advantage.

This method eventually produces a polynomial in x_5 alone, viz.

$$x_5^{26} + 245x_5^{21} + 14885x_5^{16} - 14885x_5^{11} - 245x_5^6 - x_5 =$$
$$x_5(x_5-1)(x_5^2+3x_5+1)^2(x_5^4-4x_5^3+6x_5^2+x_5+1)^2(x_5^4+x_5^3+6x_5^2-4x_5+1)^2(x_5^4+x_5^3+x_5^2+x_5+1),$$

which is something the unmodified code never did, and which this did as the 215th. polynomial. The high-water marks up to this point are: 49 remaining critical pairs, maximum total degree 23, and 14-digit coefficients, with a largest single computation time of 59 seconds. Almost immediately thereafter, the program runs out of space in $6\frac{1}{2}$ Megabytes, but has proved that there are only a finite number of solutions in x_5 (and hence, by symmetry, that there are only a finite number of solutions overall).

Factored

Observing the factorisation of the polynomial given in the previous section, we can see that their is a fair amount of redundancy in terms of repeated factors. In fact, though this polynomial is of degree 26, its square-free part is only of degree 14. Furthermore, the factor of x_5 is essentially spurious, since a local analysis at this point will reveal trivially that there are no (finite) solutions of the system of equations here. Hence we might begin to wonder whether we would be better off factoring the elements of our basis, and thus working in (an approximation to — in the sense that we may not have found all the factorisation possible) the prime decomposition of our ideal, rather than the ideal itself. The existing code supported* such an option, but only to one level, i.e., as soon as one factorisation is non-trivial (apart from discarding multiple factors), the system no longer factorises. Since this hardly met our desiderata, we modified the system to permit any number of factorisations.

This modification was a success beyond our wildest hopes: a complete answer was printed out in five minutes. This method will clearly make at least as much progress as the previous one would, but in fact it makes much more. The previous one first made a simplification at polynomial 14: the present one observes that polynomial 12 has a factor of $x_4^2 - 2x_5^2$, as well as the remaining factor of degree 5 (which has 19 terms as opposed to the 30 terms of the original polynomial). The next section describes a partial "factorisation tree" for this problem.

There were many fewer critical pairs generated or stored: the maximum number remaining at any one time being 15 (as opposed to 68 or 49 for the previous methods). Observing the system as it ran, we saw that "solutions" such as x_5 were rapidly being eliminated, and that the general factored shape of the solution was very helpful. In fact, the solution produced was

$$\begin{cases} x_5^4 - 4x_5^3 + 6x_5^2 + x_5 + 1, 11x_4 - 9x_5^3 + 39x_5^2 - 67x_5 + 6, 11x_3 + 3x_5^3 - 13x_5^2 + 26x_5 - 2, \\ 11x_2 + 3x_5^3 - 13x_5^2 + 26x_5 - 2, 11x_1 + 3x_5^3 - 13x_5^2 + 26x_5 - 2 \end{cases} \tag{1}$$

$$\begin{cases} x_5^4 + x_5^3 + 6x_5^2 - 4x_5 + 1, 11x_4 - 9x_5^3 - 15x_5^2 - 64x_5 - 3, 11x_3 + 3x_5^3 + 5x_5^2 + 25x_5 + 1, \\ 11x_2 + 3x_5^3 + 5x_5^2 + 25x_5 + 1, 11x_1 + 3x_5^3 + 5x_5^2 + 25x_5 + 1 \end{cases} \tag{2}$$

$$x_5^2 + 3x_5 + 1, x_4 + x_5 + 3, x_3 - 1, x_2 - 1, x_1 - 1 \tag{3}$$

$$\begin{cases} x_5^4 - 4x_5^3 + 6x_5^2 + x_5 + 1, 55x_4 + 15x_5^3 - 65x_5^2 + 130x_5 - 10, 11x_3 + 3x_5^3 - 13x_5^2 + 26x_5 - 2, \\ 11x_2 + 3x_5^3 - 13x_5^2 + 26x_5 - 2, 11x_1 - 9x_5^3 + 39x_5^2 - 67x_5 + 6 \end{cases} \tag{4}$$

$$\begin{cases} x_5^4 + x_5^3 + 6x_5^2 - 4x_5 + 1, 55x_4 + 15x_5^3 + 25x_5^2 + 125x_5 + 5, 11x_3 + 3x_5^3 + 5x_5^2 + 25x_5 + 1, \\ 11x_2 + 3x_5^3 + 5x_5^2 + 25x_5 + 1, 11x_1 - 9x_5^3 - 15x_5^2 - 64x_5 - 3 \end{cases} \tag{5}$$

$$x_5^2 + 3x_5 + 1, 55x_4 - 55, x_3 - 1, x_2 - 1, x_1 + x_5 + 3 \tag{6}$$

$$x_5^4 + x_5^3 + x_5^2 + x_5 + 1, x_4^2 + 3x_4x_5 + x_5^2, x_3 + x_4 + 3x_5, x_2 - x_5, x_1 - x_5 \tag{7}$$

$$x_5 - 1, x_4^2 + 3x_4 + 1, x_3 + x_4 + 3, x_2 - 1, x_1 - 1 \tag{8}$$

$$x_5^4 + x_5^3 + x_5^2 + x_5 + 1, x_4 - x_5, x_3^2 + 3x_3x_5 + x_5^2, x_2 + x_3 + 3x_5, x_1 - x_5 \tag{9}$$

$$x_5 - 1, x_4 - 1, x_3^2 + 3x_3 + 1, x_2 + x_3 + 3, x_1 - 1 \tag{10}$$

$$\begin{cases} x_5^4 + x_5^3 + x_5^2 + x_5 + 1, x_4^4 + x_4^3x_5 + x_4^2x_5^2 + x_4x_5^3 - x_5^3 - x_5^2 - x_5 - 1, \\ x_3 + x_4^2x_5^3 + x_4^2x_5^2 + x_4^2x_5 + x_4^2, x_2 - x_4^3x_5^3, x_1 + x_4^3x_5^3 - x_4^2x_5^3 - x_4^2x_5^2 - x_4^2x_5 - x_4^2 + x_4 + x_5 \end{cases} \tag{11}$$

$$x_5 - 1, x_4^4 + x_4^3 + x_4^2 + x_4 + 1, x_3 - x_4^2, x_2 - x_4^3, x_1 + x_4^3 + x_4^2 + x_4 + 1 \tag{12}$$

$$x_5^4 + x_5^3 + x_5^2 + x_5 + 1, x_4 - x_5, x_3 - x_5, x_2^2 + 3x_2x_5 + x_5^2, x_1 + x_2 + 3x_5 \tag{13}$$

$$x_5 - 1, x_4 - 1, x_3 - 1, x_2^2 + 3x_2 + 1, x_1 + x_2 + 3 \tag{14}$$

Tree-prune

This variant was based on observing that the previous algorithm often went down the same dead-end several times. Suppose, for example, that an H-polynomial factors as x_5p. We first explore the system consisting of all previous polynomials and x_5, and this eventually leads to a contradiction. Hence we explore the branch consisting of all previous polynomials and p. At some point in the future, this might (and indeed very often does) give rise to a factor which was also divisible by x_5, and we would create a new sub-tree to explore. But in fact this sub-tree is unnecessary: we have already shown that x_5 does not give rise to any solutions (or, in a more general case, we would have already discovered what the solutions were). Hence we modified the previous algorithm to keep a list of branches that had already been explored, and the algorithm would not re-descend such branches. This greatly increased the readability of the output, and also led to a substantial speed-up, as we see in the table of timings below.

* The authors have also been developing their version – see Melenk *et al.* [1987].

$$
\begin{cases}
P_2 \to \begin{cases}
x_5^3 \to \begin{cases} x_4 \to \text{impossible} \\ x_3^4 \to \text{impossible} \end{cases} \\
P_4 \to \begin{cases}
x_5 \to \text{redundant} \\
P_3 \to P_6 \to \begin{cases}
x_5^2 \to \text{redundant} \\
P_5 \to P_4 \to P_5 \to \begin{cases}
x_5^2 \to \text{redundant} \\
P_4 \to \begin{cases}
x_5^2 & \to \text{redundant} \\
(x_4 - 3x_5) & \to \text{impossible} \\
(x_3 - 2x_5) & \to \begin{cases} x_5^3 & \to \text{redundant} \\ x_4 - \frac{26}{11}x_5 & \to \text{impossible} \end{cases} \\
(x_3^2 + \cdots) & \to \begin{cases} x_5^2 & \to \text{redundant} \\ (x_3x_4 + \cdots) & \to \text{3 steps to} \\ & \to \text{impossibility} \end{cases}
\end{cases}
\end{cases}
\end{cases}
\end{cases}
\end{cases} \\
P_5 \to P_4 \to P_5 \to P_6 \to \begin{cases}
x_5^2 \to \begin{cases} x_4 & \to \text{impossible} \\ x_3 & \to \text{impossible} \\ x_3 - 2x_4 & \to \text{impossible} \end{cases} \\
P_5 \to P_6 \to P_7 \to \begin{cases}
x_4^2 & \to \begin{cases}
x_5 & \to \text{redundant} \\
x_3 + x_5 & \to \begin{cases} x_5 \to \text{redundant} \\ x_2 \to \text{impossible} \end{cases} \\
x_3 & \to \begin{cases} x_5^5 & \to \text{redundant} \\ x_2 + x_5 & \to \text{impossible} \end{cases}
\end{cases} \\
x_3 - x_4^2 & \to \text{This one leads somewhere} \\
x_4^2 - x_3 + 1 & \to \text{This one leads somewhere} \\
x_3^2 + 3x_3 + 1 & \to \text{This one leads somewhere}
\end{cases}
\end{cases}
\end{cases}
$$

The figure below shows the “tree of factorisations” that results from the splitting of the 12th. polynomial: note the frequency of the description “redundant”, which means that the improvement described in this section has truncated the tree at this point. The notation P_i stands for a polynomial of degree i.

Better

This was a mild variant on the previous algorithm, based on an apparently trivial fact about our implementation that turns out to have serious consequences. The REDUCE factoriser is written using REDUCE’s standard representation for polynomials, viz. a *recursive* representation. The Gröbner-base package, though, uses a *distributed* representation, as do most such programs. Hence it is necessary to convert between the two when calling the factoriser on the H-polynomials. Although factorisation has certainly been a great help to us, it is still the case that the majority of polynomials do not factorise, and hence the re-conversion of the output of the factoriser back into the distributed form is unnecessary: we can use the un-converted input. This gives rise to quite a substantial saving which, because of the structure of the code, is split between that part which is charged to “factorisation” and that part which is “general”.

Dip2sf

The previous improvement pointed out the great significance of representation conversion in the cost of the whole computation. We therefore investigated the techniques used a little more closely. The code as supplied converted from distributed (*dip*) to recursive (*sf* or *f*) representation via the sequence `!*q2f simp dip2a h`, i.e. conversion from distributed to prefix (*a*), thence to recursive rational functions (standard quotient or *q*), and taking the numerator to give a polynomial. This seems a little unnecessary, since we could convert directly from distributed to recursive, especially as the order of variables in the recursive representation is for us to determine, and so can be taken to be the same as that in the distributed representation.

Hence we wrote a direct conversion routine, and replaced the sequence above by `dip2sf h`. This showed a substantial saving, nearly all on the factorisation time, and reminded us how expensive representation conversion can be.

Sf2dip

This was inspired by the same ideas as above, and let us do the conversion from recursive form to distributed form (as required when the factoriser has produced a non-trivial factorisation) directly, rather than via prefix form. Fortunately, it was not necessary for us to write this code, since it was already available on the REDUCE distribution tape we had, though not incorporated into the system. This time the gains are more modest, since most of the time the factoriser discovers that the polynomials are irreducible, and the improvement described under "better" has already eliminated the conversion.

Homog

We noticed that many of the polynomials that were being factorised, at least in the early stges of the calculations, were homogeneous. This should not be totally surprising, since the only inhomogeneous polynomial is also the polynomial of highest degree, and hence is used as sparingly as possible in the algorithm. REDUCE does not possess the equivalent of MACSYMA's (undocumented) `homoghack` switch setting, i.e. the factorisation of homogeneous polynomials is not treated specially.

This is a pity, since the factorisation of a homogeneous polynomial is directly related to that of the corresponding inhomogeous polynomial with one fewer variables: just homogenise the factors, and add an additional factor of a power of the suppressed variable, if this is necessary to preserve the total degree of the factors. We added such a pre-pass to the Gröbner–factorisation interface. This had a significant effect on the factorisation times, e.g. 1564ms became 731ms, and at one point 10336ms became 901ms (but the garbage collection time has somewhat distorted this).

Homog+

This (rather minor) improvement was in fact discovered later, but it fits in the logical sequence here. In order to access individual variables for the homogeneity testing etc., we needed to use the function `evnth` for finding the n-th element of an exponent vector*. Unfortunately, we observed that this was coded using functions to access the components of the exponents, not macros, and that it was using generic arithmetic to count down the list, rather than "fixnum" arithmetic. We changed these two, and re-built the world as appropriate. The savings are modest, but real.

Results for Five variables				
Method Name	Total time	Factor time	memory	result
original	$>$ 2 hrs.	0	$6\frac{1}{2}$	fail
monomials	$>$ 2 hrs.	0	$6\frac{1}{2}$	just
factored	c. 5 mins.	c. 5 mins.	$4\frac{1}{2}$	fine
tree-prune	272.9 secs.	239.8 secs.	$4\frac{1}{2}$	fine
better	238.6 secs.	222.8 secs.	$4\frac{1}{2}$	fine
dip2sf	181.5 secs.	168.9 secs.	$4\frac{1}{2}$	fine
& sf2dip	175.7 secs.	151.9 secs.	$4\frac{1}{2}$	fine
homog	99.6 secs.	76.5 secs.	$4\frac{1}{2}$	fine
homog+	95.9 secs.	72.1 secs.	$4\frac{1}{2}$	fine

* This is the terminology always used. In fact, it is stored as a list in the standard REDUCE implementation, though Melenk *et al.* use a different representation.

Investigations in Six Variables

At this point, we felt fairly contented with ourselves, and looked at the case of six variables. A run of the "better" program had been cancelled after not achieving very much, but it had used 2911 seconds inside the factoriser, most of which went in the last few factorisation before it was cancelled: the last few being 516, 668 and 881 seconds. We decided to pursue our investigations on the computation up to this point: if we couldn't improve on this part, then there was little hope for the rest.

To get to the same point, the "homog" program used 381 seconds, which seems to indicate the improvements made above have a disproportionate effect as the number of variables increases, which one might expect, since the complexity of Gröbner-base calculations and of factoring increase steeply with the number of variables. This direct attempt on the six-variables problem rapidly showed that we were spending a great deal of our time factoring polynomials which turned out to be irreducible: indeed the factoring was taking far longer than the actual computation of the polynomial concerned, even with the improvements listed.

Sqfr

This algorithm consisted the idea of "probabilistic reduction" in order to determine whether the polynomial to be factored might be square-free. If so, the REDUCE factoriser was instructed* not to test its input polynomial for being square-free, but rather to assume that it was. In fact, we also used the same reduction to detect monomial factors directly, and perhaps to rule out low-degree (two or three) polynomials if their images had no linear factors in the reduction. This has a slight effect on the total time for the five-variable case, as can be seen in the second table, but the distribution is far more intesting: we almost halve the factorisation time as a result, even though the modular calculations are quite expensive, and eat up most of the saving.

For the (partial) computations with six variables, the saving obtained by knowing that the polynomials are square-free is definitely greater than the cost of the modular computation. In fact, we are seeing a saving of over $\frac{1}{3}$ in the total time for factorisation.

One-prime

Here we take the logical next step: having obtained this modular image, we can deduce that, if it is irreducible, then the original polynomial is irreducible. For the case of five variables, this is definitely not worth it: for the case of six variables the increase in time is less than 1%, and this is well within the errors of the timing process.

Two-evaluations

This solution consisted of using the factorisations of two reductions, not one, and returning "irreducible" if either of them said so, but without attempting an analysis of the various degrees concerned to try to deduce irreducibility from the incompatibility of the degrees of the various factors. Occasionally the second prime gave more information than the first, but this was rare. For example, H_8, a polynomial of degree four in four variables with fifteen terms, split (3,1) for the first reduction, but the second proved that it was irreducible. Most of the time in the five-variable case, though, the second factorisation was not much different from the first.

Full-modular

We next took a leaf out of Musser's book, and used several primes, and combined the modular information in order to deduce as much information as possible. For example, a (3,1) split modulo one prime, and a (2,2) split modulo another, would lead to a deduction of irreducibility. Musser [1978] suggests using five primes: we in fact use up to 10. This is not a carefully researched number: the reason we use more than Musser is that we have to deal with two additional kinds of "bad reduction" which Musser did not consider:

a) the original multivariate polynomial might acquire suprious factorisations in the process of being transformed into a univariate polynomial;
b) the modular univariate image might fail to be square-free, even though the original multivariate was square-free.

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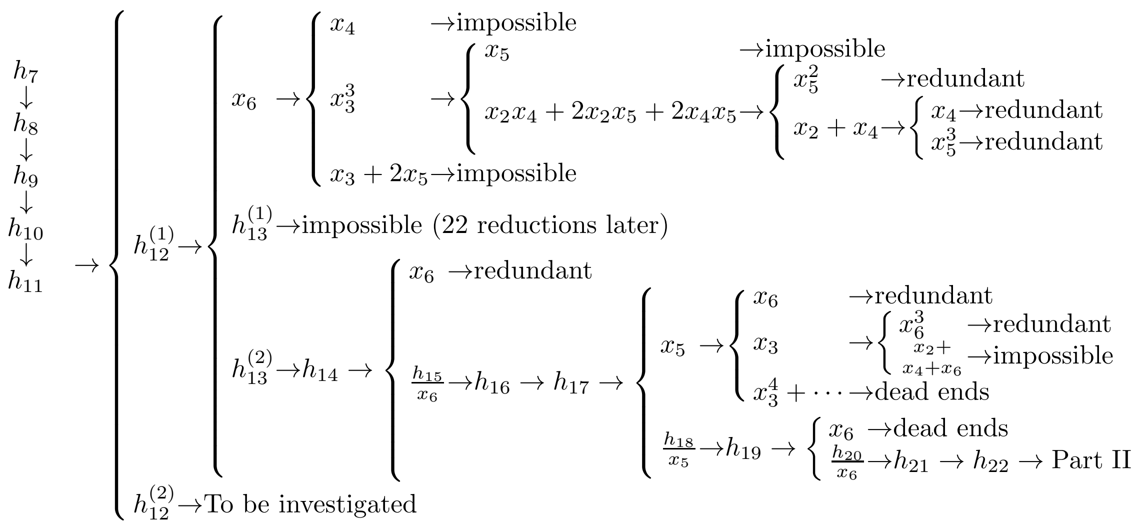
* Initially by a suitable interchange of function definitions, later on (see "Integrated") by a revision of the interface to the factoriser.

For the five-variable problem, the reduction in factorisation time is more than swallowed up by the additional costs of the extra modular reductions.

Integrated

We then integrated the modular component more into the REDUCE factoriser. Instead of swapping function definitions, we added an extra parameter to the appropriate routines, that described the known information about the factorisation. This had several advantages: the obvious one was that it was often possible to stop the REDUCE factoriser early, when content detection had found as many factors as the modular analysis had led us to predict.

Results for Five- and Part of Six-variable calculation						
	Five variables			Six variables		
Method Name	Total time	Factor time	Modular time	Total time	Factor time	Modular time
better	238.6 secs.	222.8 secs.		—	2911 secs.	
homog	99.6 secs.	76.5 secs.		—	381 secs.	
sqfr	88.2 secs.	36.2 secs.	24.9 secs.	—	200 secs.	51 secs.
one-prime	90.2 secs.	33.1 secs.	33.5 secs.	—	169 secs.	84 secs.
two-evals	107.3 secs.	31.0 secs.	51.3 secs.	—	153 secs.	152 secs.
full modular	172.0 secs.	18.6 secs.	127.6 secs.	744 secs.	41 secs.	316 secs.
integrated	173.8 secs.	19.1 secs.	127.3 secs.	734 secs.	40 secs.	306 secs.



Conclusions

Factorisation is a very powerful tool in the decomposition of large Gröbner bases, *where it is applicable* (which automatically excludes cases "in generic position"). Nevertheless, even in an example which factorises well, most of the H-polynomials will be irreducible, so it is important that such cases are easily detected.

Part II

$$h_{22} \to \begin{cases} x_5 - 2x_6 \to \begin{cases} x_6^4 & \to \text{redundant} \\ x_4 & \to \begin{cases} x_6^2 & \to \text{redundant} \\ x_3^2 - 15x_6^2 & \to \text{dead ends} \\ x_3^2 + 2x_3x_6 - 2x_6^2 & \to \text{dead ends} \end{cases} \\ x_3^3 + 5x_3^2x_6 - 2x_3x_6^2 - x_6^3 & \to \begin{cases} x_6^2 & \to \text{redundant} \\ x_3^2 - 15x_6^2 & \to \text{dead ends} \\ x_3^2 + 2x_3x_6 - 2x_6^2 & \to \text{dead ends} \end{cases} \end{cases} \\ \frac{h_{23}}{x_5 - 2x_6} \to h_{24} \to \cdots \to h_{30} \to \begin{cases} x_5^2 + 2x_5x_6 - 2x_6^2 \to \text{dead end (18 steps later)} \\ \frac{h_{31}}{x_5^2 + 2x_5x_6 - 2x_6^2} \to \cdots \to \begin{cases} x_5 - 2x_6 & \to \text{redundant} \\ \left(h_{45}^{(1)}\right)^2 & \to \text{dead end (40 steps)} \\ h_{45}^{(2)} & \to \text{being investigated} \end{cases} \end{cases} \end{cases}$$

$$h_{45}^{(1)} = x_5^6 + 2x_5^5x_6 + 5x_5^4x_6^2 + 15x_5^3x_6^3 - 192x_5^2x_6^4 + 242x_5x_6^5 - 104x_6^6$$

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Appendix

Here we give some of the polynomials that were calculated in the early stages of the six-variable calculation, to give the reader some feel for the size of polynomial, and the importance of factorisation. The polynomials are available in machine-readable form from the author.

$$x_2^2 + x_2x_4 + x_2x_5 + 2x_2x_6 - x_3x_4 + x_3x_6 - x_4x_5 + x_4x_6 + x_6^2 \tag{h_7}$$

$$\begin{gathered} x_2x_3^2 - x_2x_3x_4 + x_2x_5x_6 - x_2x_6^2 + x_3^2x_4 - x_3^2x_6 + \\ x_3x_5x_6 - 2x_3x_6^2 + x_4x_5x_6 - x_4x_6^2 + x_5^2x_6 + x_5x_6^2 - x_6^3 \end{gathered} \tag{h_8}$$

$$\begin{gathered} x_2x_3x_4^2 - x_2x_3x_4x_6 + x_2x_3x_5x_6 - 2x_2x_3x_6^2 + x_2x_5^2x_6 + 2x_2x_5x_6^2 - 2x_2x_6^3 + x_3^3x_4 - x_3^3x_6 - x_3^2x_4^2 + \\ 2x_3^2x_4x_5 + 3x_3^2x_4x_6 - x_3^2x_5x_6 - 4x_3^2x_6^2 - x_3x_4^2x_5 + x_3x_4^2x_6 + x_3x_4x_5x_6 + x_3x_5^2x_6 - 5x_3x_6^3 + 2x_4x_5^2x_6 - \\ 2x_4x_6^3 + x_5^3x_6 + 4x_5^2x_6^2 + x_5x_6^3 - 2x_6^4 \end{gathered} \tag{h_9}$$

$$\begin{gathered} x_2x_3x_5^2x_6 + 2x_2x_3x_5x_6^2 - 2x_2x_3x_6^3 - x_2x_4^2x_5x_6 + x_2x_4^2x_6^2 - x_2x_4x_5^2x_6 - x_2x_4x_5x_6^2 + x_2x_4x_6^3 - x_2x_5^2x_6^2 + \\ 3x_2x_5x_6^3 - 2x_2x_6^4 + x_3^4x_4 - x_3^4x_6 - 2x_3^3x_4^2 + 2x_3^3x_4x_5 + 4x_3^3x_4x_6 - x_3^3x_5x_6 - 4x_3^3x_6^2 - 3x_3^2x_4^2x_5 + x_3^2x_4x_5x_6 + \\ 5x_3^2x_4x_6^2 + x_3^2x_5^2x_6 + x_3^2x_5x_6^2 - 7x_3^2x_6^3 + x_3x_4^3x_5 - x_3x_4^3x_6 - 2x_3x_4^2x_5x_6 + 2x_3x_4^2x_6^2 + x_3x_4x_5^2x_6 + x_3x_4x_5x_6^2 + \\ x_3x_4x_6^3 + x_3x_5^3x_6 + 3x_3x_5^2x_6^2 + 5x_3x_5x_6^3 - 6x_3x_6^4 - x_4^3x_5x_6 + x_4^3x_6^2 - 3x_4^2x_5^2x_6 + 2x_4^2x_6^3 - x_4x_5^3x_6 - \\ 4x_4x_5^2x_6^2 + 3x_4x_5x_6^3 - x_4x_6^4 - x_5^3x_6^2 + x_5^2x_6^3 + 3x_5x_6^4 - 2x_6^5 \end{gathered} \tag{h_{10}}$$

$$x_2x_4^4x_5x_6 - x_2x_4^4x_6^2 + x_2x_4^3x_5^2x_6 + x_2x_4^2x_5^2x_6^2 - 7x_2x_4^2x_5x_6^3 + 5x_2x_4^2x_6^4 + x_2x_4x_5^3x_6^2 - 2x_2x_4x_5^2x_6^3 + x_2x_5^4x_6^2+$$
$$5x_2x_5^3x_6^3 - 5x_2x_5^2x_6^4 - x_3^4x_4^3 + 2x_3^4x_4^2x_6 - x_3^4x_4x_5x_6 + x_3^4x_4x_6^2 + x_3^4x_5x_6^2 - 2x_3^4x_6^3 + 2x_3^3x_4^4 - 2x_3^3x_4^3x_5$$
$$-6x_3^3x_4^3x_6 + 5x_3^3x_4^2x_5x_6 + 4x_3^3x_4^2x_6^2 - x_3^3x_4x_5^2x_6 + x_3^3x_4x_5x_6^2 + 2x_3^3x_4x_6^3 - 6x_3^3x_6^4 + 3x_3^2x_4^4x_5 - 4x_3^2x_4^3x_5x_6-$$
$$5x_3^2x_4^3x_6^2 + x_3^2x_4^2x_5^2x_6 - 8x_3^2x_4^2x_5x_6^2 + 14x_3^2x_4^2x_6^3 + 2x_3^2x_4x_5^3x_6 + 7x_3^2x_4x_5^2x_6^2 - 3x_3^2x_4x_6^4 - 2x_3^2x_5^3x_6^2-$$
$$5x_3^2x_5^2x_6^3 + 3x_3^2x_5x_6^4 - 6x_3^2x_6^5 - x_3x_4^5x_5 + x_3x_4^5x_6 + 3x_3x_4^4x_5x_6 - 3x_3x_4^4x_6^2 - 2x_3x_4^3x_5^2x_6 - x_3x_4^3x_6^3-$$
$$2x_3x_4^2x_5^3x_6 - x_3x_4^2x_5^2x_6^2 - 6x_3x_4^2x_5x_6^3 + 9x_3x_4^2x_6^4 + x_3x_4x_5^3x_6^2 + 6x_3x_4x_5^2x_6^3 + 4x_3x_4x_5x_6^4 - 4x_3x_4x_6^5 + x_3x_5^3x_6^3$$
$$-6x_3x_5^2x_6^4 + 6x_3x_5x_6^5 - 2x_3x_6^6 + x_4^5x_5x_6 - x_4^5x_6^2 + 3x_4^4x_5^2x_6 - x_4^4x_5x_6^2 - x_4^4x_6^3 + x_4^3x_5^3x_6 + 2x_4^3x_5^2x_6^2 - 6x_4^3x_5x_6^3+$$
$$5x_4^3x_6^4 + 3x_4^2x_5^3x_6^2 - 11x_4^2x_5^2x_6^3 - 2x_4^2x_5x_6^4 + 5x_4^2x_6^5 + 3x_4x_5^4x_6^2 + 5x_4x_5^3x_6^3 - 16x_4x_5^2x_6^4 + 6x_4x_5x_6^5 + x_5^5x_6^2+$$
$$7x_5x_4x_5^3x_6^3 - 16x_4x_5^2x_6^4 + 6x_4x_5x_6^5 + x_5^5x_6^2 + 7x_5^4x_6^3 + 4x_5^3x_6^4 - 9x_5^2x_6^5 + 2x_5x_6^6 \qquad (h_{11})$$

$$x_3^5x_4^3 - 2x_3^5x_4^2x_6 + x_3^5x_4x_5x_6 - x_3^5x_4x_6^2 - x_3^5x_5x_6^2 + 2x_3^5x_6^3 - 2x_3^4x_4^4 + 2x_3^4x_4^3x_5 + 6x_3^4x_4^3x_6 - 5x_3^4x_4^2x_5x_6-$$
$$4x_3^4x_4^2x_6^2 + 2x_3^4x_4x_5^2x_6 + x_3^4x_4x_5x_6^2 - 4x_3^4x_4x_6^3 - x_3^4x_5^2x_6^2 - 2x_3^4x_5x_6^3 + 8x_3^4x_6^4 - 3x_3^3x_4^4x_5 + 5x_3^3x_4^3x_5x_6+$$
$$4x_3^3x_4^3x_6^2 - 2x_3^3x_4^2x_5^2x_6 + 4x_3^3x_4^2x_5x_6^2 - 10x_3^3x_4^2x_6^3 + x_3^3x_4x_5^2x_6^2 - 2x_3^3x_4x_6^4 + x_3^3x_5^3x_6^2 - 2x_3^3x_5^2x_6^3 - 6x_3^3x_5x_6^4+$$
$$12x_3^3x_6^5 + x_3^2x_4^5x_5 - x_3^2x_4^5x_6 - 4x_3^2x_4^4x_5x_6 + 4x_3^2x_4^4x_6^2 + 3x_3^2x_4^3x_5^2x_6 - x_3^2x_4^3x_6^3 + x_3^2x_4^2x_5^3x_6 - 2x_3^2x_4^2x_5^2x_6^2+$$
$$13x_3^2x_4^2x_5x_6^3 - 10x_3^2x_4^2x_6^4 + x_3^2x_4x_5^3x_6^2 - 7x_3^2x_4x_5^2x_6^3 - 4x_3^2x_4x_5x_6^4 + 4x_3^2x_4x_6^5 + x_3^2x_5^4x_6^2 + x_3^2x_5^3x_6^3-$$
$$2x_3^2x_5^2x_6^4 - 12x_3^2x_5x_6^5 + 8x_3^2x_6^6 - x_3x_4^5x_5x_6 + x_3x_4^5x_6^2 - 4x_3x_4^4x_5^2x_6 + 3x_3x_4^4x_5x_6^2 - x_3x_4^3x_5^3x_6 + 3x_3x_4^3x_5x_6^3-$$
$$4x_3x_4^3x_6^4 - 4x_3x_4^2x_5^3x_6^2 + 13x_3x_4^2x_5^2x_6^3 - x_3x_4^2x_5x_6^4 - 2x_3x_4^2x_6^5 - x_3x_4x_5^4x_6^2 + 8x_3x_4x_5^2x_6^4 - 9x_3x_4x_5x_6^5$$
$$+3x_3x_4x_6^6 - x_3x_5^4x_6^3 + 2x_3x_5^3x_6^4 + 4x_3x_5^2x_6^5 - 9x_3x_5x_6^6 + 2x_3x_6^7 + x_4^3x_5^3x_6^2 - 3x_4^3x_5^2x_6^3 + 2x_4^3x_5x_6^4 - x_4^2x_5^3x_6^3+$$
$$x_4^2x_5^2x_6^4 - x_4^2x_5x_6^5 + x_4x_5^4x_6^3 - 5x_4x_5^3x_6^4 + 10x_4x_5^2x_6^5 - 5x_4x_5x_6^6 - 2x_5^3x_6^5 + 5x_5^2x_6^6 - 2x_5x_6^7 \qquad (h_{12})$$

$$h_{12} = h_{12}^{(1)}h_{12}^{(2)}$$
$$x_4^2 - x_4x_6 - 2x_6^2 + x_6x_5 \qquad (h_{12}^{(1)})$$
$$x_3^5x_4 - x_3^5x_6 - 2x_3^4x_4^2 + 4x_3^4x_4x_6 + 2x_3^4x_4x_5 - 4x_3^4x_6^2 - x_3^4x_6x_5 - 3x_3^3x_4^2x_5 + 4x_3^3x_4x_6^2 + 2x_3^3x_4x_6x_5-$$
$$6x_3^3x_6^3 + x_3^3x_6x_5^2 - x_3^2x_4^3x_6 + x_3^2x_4^3x_5 + 3x_3^2x_4^2x_6^2 - 3x_3^2x_4^2x_6x_5 + 2x_3^2x_4x_6x_5^2 - 4x_3^2x_6^4 + 4x_3^2x_6^3x_5 + 3x_3^2x_6^2x_5^2+$$
$$x_3^2x_6x_5^3 + x_3x_4^3x_6^2 - x_3x_4^3x_6x_5 + x_3x_4^2x_6^3 + 2x_3x_4^2x_6^2x_5 - 4x_3x_4^2x_6x_5^2 - x_3x_4x_6^4 + 2x_3x_4x_6^3x_5 - 3x_3x_4x_6^2x_5^2-$$
$$x_3x_4x_6x_5^3 - x_3x_6^5 + 4x_3x_6^4x_5 - x_3x_6^2x_5^3 + 2x_4x_6^4x_5 - 3x_4x_6^3x_5^2 + x_4x_6^2x_5^3 + x_6^5x_5 - 2x_6^4x_5^2 \qquad (h_{12}^{(2)})$$

$$x_2x_5^4x_6^2 + 4x_2x_5^3x_6^3 - 8x_2x_5x_6^5 + 4x_2x_6^6 + x_3^3x_4x_5^2x_6 + 2x_3^3x_4x_5x_6^2 - 2x_3^3x_4x_6^3 - x_3^3x_5^2x_6^2 - 2x_3^3x_5x_6^3+$$
$$2x_3^3x_6^4 + 2x_3^2x_4x_5^3x_6 + 6x_3^2x_4x_5^2x_6^2 - 4x_3^2x_4x_6^4 - 6x_3^2x_5^2x_6^3 - 12x_3^2x_5x_6^4 + 12x_3^2x_6^5 + x_3x_4x_5^2x_6^3 + 2x_3x_4x_5x_6^4-$$
$$2x_3x_4x_6^5 + 2x_3x_5^4x_6^2 + x_3x_5^3x_6^3 - 13x_3x_5^2x_6^4 + 6x_3x_6^6 + 2x_4x_5^4x_6^2 + 4x_4x_5^3x_6^3 - 6x_4x_5^2x_6^4 - 4x_4x_5x_6^5 + 4x_4x_6^6+$$
$$x_5^5x_6^2 + 6x_5^4x_6^3 + 7x_5^3x_6^4 - 8x_5^2x_6^5 - 6x_5x_6^6 + 4x_6^7 \qquad (h_{13})$$

$$h_{13} = x_6h_{13}^{(1)}h_{13}^{(2)}$$
$$2x_6^2 - 2x_6x_5 - x_5^2 \qquad (h_{13}^{(1)})$$
$$2x_2x_6^3 - 2x_2x_6^2x_5 - x_2x_6x_5^2 - x_3^3x_4 + x_3^3x_6 - 2x_3^2x_4x_6 - 2x_3^2x_4x_5 + 6x_3^2x_6^2 - x_3x_4x_6^2 + 3x_3x_6^3+$$
$$3x_3x_6^2x_5 - 2x_3x_6x_5^2 + 2x_4x_6^3 - 2x_4x_6x_5^2 + 2x_6^4 - x_6^3x_5 - 4x_6^2x_5^2 - x_6x_5^3 \qquad (h_{13}^{(2)})$$

$$x_3^5x_4 - x_3^5x_6 + 2x_3^4x_4x_5 + 2x_3^4x_4x_6 + x_3^4x_5x_6 - 8x_3^4x_6^2 - x_3^3x_4x_5x_6 + 4x_3^3x_4x_6^2 + 4x_3^3x_5^2x_6 - 6x_3^3x_5x_6^2 - 6x_3^3x_6^3+$$
$$x_3^2x_4x_5^2x_6 + x_3^2x_4x_5x_6^2 + x_3^2x_5^3x_6 + 5x_3^2x_5^2x_6^2 - 2x_3^2x_5x_6^3 - x_3x_4x_5^3x_6 - 6x_3x_4x_5^2x_6^2 + 3x_3x_4x_6^4 + 3x_3x_5^3x_6^2-$$
$$9x_3x_5^2x_6^3 + 4x_3x_5x_6^4 + 3x_3x_6^5 + x_4x_5^3x_6^2 - 3x_4x_5^2x_6^3 + 2x_4x_5x_6^4 - 2x_5^2x_6^4 + x_5x_6^5 \qquad (h_{14})$$

$$x_3^5x_5x_6 - 2x_3^5x_6^2 - 3x_3^4x_4x_5x_6 + 6x_3^4x_4x_6^2 + 2x_3^4x_5^2x_6 - 2x_3^4x_5x_6^2 - 4x_3^4x_6^3 - 4x_3^3x_4x_5^2x_6 + 7x_3^3x_4x_5x_6^2 + 2x_3^3x_4x_6^3 - x_3^3x_5^2x_6^2 + 6x_3^3x_5x_6^3 - 8x_3^3x_6^4 - x_3^2x_4x_5^3x_6 - 6x_3^2x_4x_5^2x_6^2 + x_3^2x_4x_5x_6^3 + x_3^2x_5^3x_6^2 - x_3^2x_5^2x_6^3 - 2x_3^2x_5x_6^4 - 2x_3x_4x_5^3x_6^2 + 15x_3x_4x_5^2x_6^3 - 4x_3x_4x_5x_6^4 - 6x_3x_4x_6^5 - x_3x_5^4x_6^2 - 4x_3x_5^3x_6^3 + 12x_3x_5^2x_6^4 + 3x_3x_5x_6^5 - 6x_3x_6^6 - x_4x_5^3x_6^3 + 5x_4x_5^2x_6^4 - 3x_4x_5x_6^5 + x_5^4x_6^3 - 5x_5^3x_6^4 + 8x_5^2x_6^5 - 4x_5x_6^6 \tag{h_{15}}$$

$$x_2x_3x_4x_6 + x_2x_3x_6^2 + x_2x_4x_6^2 - 2x_2x_5x_6^2 + x_2x_6^3 + x_3^2x_4x_5 + x_3x_4x_5^2 + 5x_3x_4x_5x_6 - x_3x_4x_6^2 - 2x_3x_5^2x_6 + 3x_3x_5x_6^2 - x_3x_6^3 + x_4x_5^2x_6 - 2x_4x_5x_6^2 + 3x_4x_6^3 - x_5^2x_6^2 - 3x_5x_6^3 + 3x_6^4 \tag{h_{16}}$$

$$x_3^4x_4x_6 - x_3^4x_5x_6 + x_3^4x_6^2 + 2x_3^3x_4x_5x_6 - x_3^3x_4x_6^2 - 2x_3^3x_5^2x_6 + 3x_3^3x_5x_6^2 - x_3^3x_6^3 - x_3^2x_4x_5^3 - 4x_3^2x_4x_5^2x_6 - x_3^2x_4x_5x_6^2 - 3x_3^2x_4x_6^3 + 10x_3^2x_5x_6^3 - 3x_3^2x_6^4 - x_3x_4x_5^4 - 6x_3x_4x_5^3x_6 + 3x_3x_4x_5^2x_6^2 + 8x_3x_4x_5x_6^3 - 9x_3x_4x_6^4 + 2x_3x_5^4x_6 - 4x_3x_5^3x_6^2 + 7x_3x_5^2x_6^3 + 13x_3x_5x_6^4 - 9x_3x_6^5 - x_4x_5^4x_6 - 3x_4x_5^3x_6^2 + 11x_4x_5^2x_6^3 - 5x_4x_5x_6^4 - x_5^4x_6^2 - 4x_5^3x_6^3 + 7x_5^2x_6^4 - 5x_5x_6^5 \tag{h_{17}}$$

$$x_3^4x_5^2x_6 - 2x_3^4x_5x_6^2 + 2x_3^3x_5^3x_6 - 5x_3^3x_5^2x_6^2 + 2x_3^3x_5x_6^3 + x_3^2x_4x_5^4 + 3x_3^2x_4x_5^3x_6 - 3x_3^2x_4x_5^2x_6^2 - 8x_3^2x_4x_5x_6^3 - x_3^2x_5^4x_6 - 2x_3^2x_5^3x_6^2 - 3x_3^2x_5^2x_6^3 + 22x_3^2x_5x_6^4 + x_3x_4x_5^5 + 3x_3x_4x_5^4x_6 - 5x_3x_4x_5^3x_6^2 - 12x_3x_4x_5^2x_6^3 + 4x_3x_4x_5x_6^4 - 3x_3x_5^5x_6 + 4x_3x_5^4x_6^2 + 3x_3x_5^2x_6^4 + 10x_3x_5x_6^5 + x_4x_5^5x_6 + 3x_4x_5^4x_6^2 - 10x_4x_5^3x_6^3 + 9x_4x_5^2x_6^4 + x_5^4x_6^3 + 2x_5^3x_6^4 - 8x_5^2x_6^5 \tag{h_{18}}$$

$$x_3^3x_4x_5^3 + x_3^3x_4x_5^2x_6 + 5x_3^3x_4x_5x_6^2 - 16x_3^3x_4x_6^3 - x_3^3x_5^3x_6 + 5x_3^3x_5^2x_6^2 - 24x_3^3x_5x_6^3 + 36x_3^3x_6^4 + x_3^2x_4x_5^4 + 7x_3^2x_4x_5^3x_6 + 7x_3^2x_4x_5^2x_6^2 - 4x_3^2x_4x_5x_6^3 - 62x_3^2x_4x_6^4 + 3x_3^2x_5^3x_6^2 - 32x_3^2x_5^2x_6^3 - 10x_3^2x_5x_6^4 + 124x_3^2x_6^5 + 10x_3x_4x_5^4x_6 - 7x_3x_4x_5^3x_6^2 - 25x_3x_4x_5^2x_6^3 + 4x_3x_4x_5x_6^4 - 24x_3x_4x_6^5 + 3x_3x_5^5x_6 - 5x_3x_5^4x_6^2 - 16x_3x_5^3x_6^3 - 16x_3x_5^2x_6^4 + 82x_3x_5x_6^5 + 12x_3x_6^6 - 2x_4x_5^3x_6^3 - 11x_4x_5^2x_6^4 + 27x_4x_5x_6^5 + 3x_5^5x_6^2 + 2x_5^4x_6^3 - 40x_5^3x_6^4 + 85x_5^2x_6^5 - 74x_5x_6^6 \tag{h_{19}}$$

$$x_3^3x_4x_5^2x_6^2 - \frac{19}{6}x_3^3x_4x_5x_6^3 + \frac{10}{3}x_3^3x_4x_6^4 - \frac{1}{6}x_3^3x_5^4x_6 + \frac{1}{6}x_3^3x_5^3x_6^2 - \frac{1}{2}x_3^3x_5^2x_6^3 + \frac{13}{3}x_3^3x_5x_6^4 - \frac{16}{3}x_3^3x_6^5 + \frac{1}{6}x_3^2x_4x_5^4x_6 + \frac{5}{3}x_3^2x_4x_5^3x_6^2 - \frac{25}{6}x_3^2x_4x_5^2x_6^3 - \frac{7}{3}x_3^2x_4x_5x_6^4 + \frac{31}{3}x_3^2x_4x_6^5 - \frac{1}{6}x_3^2x_5^5x_6 - \frac{5}{6}x_3^2x_5^4x_6^2 + \frac{7}{6}x_3^2x_5^3x_6^3 + 3x_3^2x_5^2x_6^4 + 9x_3^2x_5x_6^5 - \frac{62}{3}x_3^2x_6^6 + \frac{1}{2}x_3x_4x_5^5x_6 + \frac{5}{6}x_3x_4x_5^4x_6^2 - \frac{23}{6}x_3x_4x_5^3x_6^3 - \frac{41}{6}x_3x_4x_5^2x_6^4 + \frac{43}{3}x_3x_4x_5x_6^5 - 2x_3x_4x_6^6 - \frac{5}{3}x_3x_5^5x_6^2 + \frac{9}{2}x_3x_5^4x_6^3 + \frac{11}{6}x_3x_5^3x_6^4 - 9x_3x_5^2x_6^5 + \frac{16}{3}x_3x_5x_6^6 - 8x_3x_6^7 + \frac{1}{2}x_4x_5^5x_6^2 + \frac{1}{3}x_4x_5^4x_6^3 - 7x_4x_5^3x_6^4 + \frac{37}{3}x_4x_5^2x_6^5 - \frac{47}{6}x_4x_5x_6^6 + \frac{1}{3}x_5^4x_6^4 + \frac{7}{6}x_5^3x_6^5 - \frac{49}{6}x_5^2x_6^6 + 9x_5x_6^7 \tag{h_{20}}$$

$$x_3^3x_4x_5x_6^3 - \frac{1076}{535}x_3^3x_4x_6^4 + \frac{6}{535}x_3^3x_5^5 + \frac{19}{535}x_3^3x_5^4x_6 - \frac{43}{535}x_3^3x_5^3x_6^2 + \frac{99}{535}x_3^3x_5^2x_6^3 - \frac{1322}{535}x_3^3x_5x_6^4 + \frac{2096}{535}x_3^3x_6^5 - \frac{6}{535}x_3^2x_4x_5^5 - \frac{49}{535}x_3^2x_4x_5^4x_6 + \frac{152}{535}x_3^2x_4x_5^3x_6^2 + \frac{961}{535}x_3^2x_4x_5^2x_6^3 - \frac{166}{535}x_3^2x_4x_5x_6^4 - \frac{3782}{535}x_3^2x_4x_6^5 + \frac{6}{535}x_3^2x_5^6 + \frac{11}{107}x_3^2x_5^5x_6 + \frac{83}{535}x_3^2x_5^4x_6^2 - \frac{35}{107}x_3^2x_5^3x_6^3 - \frac{18}{5}x_3^2x_5^2x_6^4 - \frac{966}{535}x_3^2x_5x_6^5 + \frac{7564}{535}x_3^2x_6^6 - \frac{18}{535}x_3x_4x_5^6 - \frac{21}{107}x_3x_4x_5^5x_6 + \frac{373}{535}x_3x_4x_5^4x_6^2 + \frac{569}{535}x_3x_4x_5^3x_6^3 - \frac{391}{535}x_3x_4x_5^2x_6^4 - \frac{1934}{535}x_3x_4x_5x_6^5 - \frac{564}{535}x_3x_4x_6^6 + \frac{12}{107}x_3x_5^6x_6 + \frac{196}{535}x_3x_5^5x_6^2 - \frac{921}{535}x_3x_5^4x_6^3 - \frac{527}{535}x_3x_5^3x_6^4 + \frac{582}{535}x_3x_5^2x_6^5 + \frac{488}{107}x_3x_5x_6^6 + \frac{1632}{535}x_3x_6^7 - \frac{18}{535}x_4x_5^6x_6 - \frac{87}{535}x_4x_5^5x_6^2 + \frac{202}{535}x_4x_5^4x_6^3 + \frac{534}{535}x_4x_5^3x_6^4 - \frac{1964}{535}x_4x_5^2x_6^5 + \frac{2147}{535}x_4x_5x_6^6 + \frac{96}{535}x_5^5x_6^3 - \frac{4}{107}x_5^4x_6^4 - \frac{1321}{535}x_5^3x_6^5 + \frac{3961}{535}x_5^2x_6^6 - \frac{4014}{535}x_5x_6^7 \tag{h_{21}}$$

$$
\begin{aligned}
&x_3^3x_4x_6^5-\frac{535}{48156}x_3^3x_5^6-\frac{269}{12039}x_3^3x_5^5x_6-\frac{2159}{48156}x_3^3x_5^4x_6^2-\frac{1769}{16052}x_3^3x_5^3x_6^3+\frac{17371}{8026}x_3^3x_5^2x_6^4-\frac{58189}{24078}x_3^3x_5x_6^5-\\
&\frac{9619}{12039}x_3^3x_6^6+\frac{535}{48156}x_3^2x_4x_5^6+\frac{3751}{48156}x_3^2x_4x_5^5x_6-\frac{10651}{48156}x_3^2x_4x_5^4x_6^2+\frac{4739}{24078}x_3^2x_4x_5^3x_6^3-\frac{21239}{12039}x_3^2x_4x_5^2x_6^4+\\
&\frac{52204}{12039}x_3^2x_4x_5x_6^5+\frac{25823}{12039}x_3^2x_4x_6^6-\frac{535}{48156}x_3^2x_5^7-\frac{2143}{24078}x_3^2x_5^6x_6-\frac{9685}{48156}x_3^2x_5^5x_6^2-\frac{7799}{24078}x_3^2x_5^4x_6^3+\\
&\frac{52340}{12039}x_3^2x_5^3x_6^4+\frac{2568}{4013}x_3^2x_5^2x_6^5-\frac{86161}{12039}x_3^2x_5x_6^6-\frac{51646}{12039}x_3^2x_6^7+\frac{535}{16052}x_3x_4x_5^7+\frac{1877}{12039}x_3x_4x_5^6x_6-\\
&\frac{20225}{48156}x_3x_4x_5^5x_6^2+\frac{9149}{16052}x_3x_4x_5^4x_6^3-\frac{29793}{16052}x_3x_4x_5^3x_6^4-\frac{96907}{24078}x_3x_4x_5^2x_6^5+\frac{267397}{24078}x_3x_4x_5x_6^6-\\
&\frac{12793}{4013}x_3x_4x_6^7-\frac{2675}{24078}x_3x_5^7x_6-\frac{3765}{16052}x_3x_5^6x_6^2+\frac{7603}{16052}x_3x_5^5x_6^3+\frac{166771}{48156}x_3x_5^4x_6^4-\frac{18733}{48156}x_3x_5^3x_6^5-\\
&\frac{293471}{24078}x_3x_5^2x_6^6+\frac{180145}{24078}x_3x_5x_6^7-\frac{17791}{4013}x_3x_6^8+\frac{535}{16052}x_4x_5^7x_6+\frac{5903}{48156}x_4x_5^6x_6^2-\frac{1041}{16052}x_4x_5^5x_6^3-\\
&\frac{1817}{8026}x_4x_5^4x_6^4-\frac{103789}{48156}x_4x_5^3x_6^5+\frac{48641}{12039}x_4x_5^2x_6^6-\frac{76241}{24078}x_4x_5x_6^7-\frac{2140}{12039}x_5^6x_6^3+\frac{5837}{24078}x_5^5x_6^4+\\
&\frac{65815}{24078}x_5^4x_6^5-\frac{216817}{24078}x_5^3x_6^6+\frac{94106}{12039}x_5^2x_6^7+\frac{3946}{12039}x_5x_6^8 \qquad (h_{22})
\end{aligned}
$$

$$
\begin{aligned}
&x_3^3x_5^7+x_3^3x_5^5x_6^2+5x_3^3x_5^4x_6^3-222x_3^3x_5^3x_6^4+626x_3^3x_5^2x_6^5-588x_3^3x_5x_6^6+208x_3^3x_6^7-x_3^2x_4x_5^7-5x_3^2x_4x_5^6x_6+\\
&33x_3^2x_4x_5^5x_6^2-66x_3^2x_4x_5^4x_6^3+220x_3^2x_4x_5^3x_6^4-548x_3^2x_4x_5^2x_6^5+564x_3^2x_4x_5x_6^6-248x_3^2x_4x_6^7+x_3^2x_5^8+\\
&6x_3^2x_5^7x_6+3x_3^2x_5^6x_6^2+2x_3^2x_5^5x_6^3-436x_3^2x_5^4x_6^4+700x_3^2x_5^3x_6^5+436x_3^2x_5^2x_6^6-1072x_3^2x_5x_6^7+496x_3^2x_6^8-\\
&3x_3x_4x_5^8-8x_3x_4x_5^7x_6+63x_3x_4x_5^6x_6^2-145x_3x_4x_5^5x_6^3+333x_3x_4x_5^4x_6^4+122x_3x_4x_5^3x_6^5-1794x_3x_4x_5^2x_6^6+\\
&1972x_3x_4x_5x_6^7-672x_3x_4x_6^8+10x_3x_5^8x_6+x_3x_5^7x_6^2-75x_3x_5^6x_6^3-193x_3x_5^5x_6^4+507x_3x_5^4x_6^5+938x_3x_5^3x_6^6-\\
&2782x_3x_5^2x_6^7+2164x_3x_5x_6^8-528x_3x_6^9-3x_4x_5^8x_6-5x_4x_5^7x_6^2+25x_4x_5^6x_6^3-6x_4x_5^5x_6^4+187x_4x_5^4x_6^5-\\
&664x_4x_5^3x_6^6+686x_4x_5^2x_6^7-212x_4x_5x_6^8+16x_5^7x_6^3-54x_5^6x_6^4-186x_5^5x_6^5+1302x_5^4x_6^6-\\
&2556x_5^3x_6^7+2052x_5^2x_6^8-616x_5x_6^9 \qquad (h_{23})
\end{aligned}
$$

$$
\begin{aligned}
&x_3^4x_6^7+\frac{1}{60}x_3^3x_5^5x_6^3-\frac{7}{30}x_3^3x_5^4x_6^4+\frac{521}{60}x_3^3x_5^3x_6^5-\frac{442}{15}x_3^3x_5^2x_6^6+\frac{197}{6}x_3^3x_5x_6^7-\frac{208}{15}x_3^3x_6^8+\frac{1}{60}x_3^2x_4x_5^8+\frac{1}{6}x_3^2x_4x_5^7x_6+\\
&\frac{2}{5}x_3^2x_4x_5^6x_6^2-\frac{59}{60}x_3^2x_4x_5^5x_6^3+\frac{49}{12}x_3^2x_4x_5^4x_6^4-\frac{527}{30}x_3^2x_4x_5^3x_6^5+\frac{129}{5}x_3^2x_4x_5^2x_6^6-\frac{84}{5}x_3^2x_4x_5x_6^7+\frac{244}{15}x_3^2x_4x_6^8-\\
&\frac{1}{30}x_3^2x_5^8x_6-\frac{3}{10}x_3^2x_5^7x_6^2-\frac{1}{4}x_3^2x_5^6x_6^3+\frac{11}{6}x_3^2x_5^5x_6^4+\frac{112}{5}x_3^2x_5^4x_6^5-\frac{247}{5}x_3^2x_5^3x_6^6-\frac{56}{3}x_3^2x_5^2x_6^7+\frac{914}{15}x_3^2x_5x_6^8-\\
&\frac{689}{15}x_3^2x_6^9+\frac{1}{60}x_3x_4x_5^9+\frac{1}{6}x_3x_4x_5^8x_6+\frac{3}{20}x_3x_4x_5^7x_6^2-\frac{25}{12}x_3x_4x_5^6x_6^3+\frac{91}{12}x_3x_4x_5^5x_6^4-\frac{821}{30}x_3x_4x_5^4x_6^5+\\
&\frac{733}{60}x_3x_4x_5^3x_6^6+\frac{1787}{15}x_3x_4x_5^2x_6^7-\frac{275}{2}x_3x_4x_5x_6^8+\frac{751}{15}x_3x_4x_6^9-\frac{1}{20}x_3x_5^9x_6-\frac{7}{20}x_3x_5^8x_6^2+\frac{1}{20}x_3x_5^7x_6^3+\\
&\frac{251}{60}x_3x_5^6x_6^4+\frac{653}{60}x_3x_5^5x_6^5-\frac{1309}{30}x_3x_5^4x_6^6-\frac{155}{4}x_3x_5^3x_6^7+178x_3x_5^2x_6^8-\frac{4973}{30}x_3x_5x_6^9+\frac{568}{15}x_3x_6^{10}+\\
&\frac{1}{60}x_4x_5^9x_6+\frac{1}{6}x_4x_5^8x_6^2+\frac{2}{15}x_4x_5^7x_6^3-\frac{1}{2}x_4x_5^6x_6^4-\frac{21}{10}x_4x_5^5x_6^5-\frac{249}{20}x_4x_5^4x_6^6+\frac{1049}{20}x_4x_5^3x_6^7-\frac{111}{2}x_4x_5^2x_6^8+\\
&\frac{85}{6}x_4x_5x_6^9-\frac{1}{30}x_5^8x_6^3-\frac{1}{2}x_5^7x_6^4+\frac{83}{20}x_5^6x_6^5+\frac{14}{3}x_5^5x_6^6-\frac{338}{5}x_5^4x_6^7+\frac{4657}{30}x_5^3x_6^8-\\
&\frac{2027}{15}x_5^2x_6^9+\frac{145}{3}x_5x_6^{10} \qquad (h_{24})
\end{aligned}
$$

$$
\begin{aligned}
&x_3^3x_5^5x_6^4-\frac{272}{9}x_3^3x_5^4x_6^5+\frac{2825}{18}x_3^3x_5^3x_6^6-\frac{2789}{9}x_3^3x_5^2x_6^7+\frac{2357}{9}x_3^3x_5x_6^8-\frac{824}{9}x_3^3x_6^9-\frac{1}{18}x_3^2x_4x_5^9-\\
&\frac{4}{9}x_3^2x_4x_5^8x_6-\frac{2}{9}x_3^2x_4x_5^7x_6^2+\frac{53}{9}x_3^2x_4x_5^6x_6^3-\frac{185}{9}x_3^2x_4x_5^5x_6^4+\frac{521}{6}x_3^2x_4x_5^4x_6^5-\frac{604}{3}x_3^2x_4x_5^3x_6^6+\frac{2224}{9}x_3^2x_4x_5^2x_6^7-
\end{aligned}
$$

$$\frac{1696}{9}x_3^2x_4x_5x_6^8+\frac{266}{3}x_3^2x_4x_6^9+\frac{1}{9}x_3^2x_5^9x_6+\frac{7}{9}x_3^2x_5^8x_6^2-\frac{10}{9}x_3^2x_5^7x_6^3-\frac{22}{3}x_3^2x_5^6x_6^4-\frac{1105}{18}x_3^2x_5^5x_6^5+\frac{2846}{9}x_3^2x_5^4x_6^6-$$
$$\frac{2612}{9}x_3^2x_5^3x_6^7-\frac{3014}{9}x_3^2x_5^2x_6^8+\frac{5150}{9}x_3^2x_5x_6^9-\frac{740}{3}x_3^2x_6^{10}-\frac{1}{18}x_3x_4x_5^{10}-\frac{4}{9}x_3x_4x_5^9x_6+\frac{11}{18}x_3x_4x_5^8x_6^2+$$
$$\frac{70}{9}x_3x_4x_5^7x_6^3-\frac{719}{18}x_3x_4x_5^6x_6^4+\frac{2587}{18}x_3x_4x_5^5x_6^5-224x_3x_4x_5^4x_6^6-\frac{591}{2}x_3x_4x_5^3x_6^7+1263x_3x_4x_5^2x_6^8-$$
$$1169x_3x_4x_5x_6^9+\frac{3292}{9}x_3x_4x_6^{10}+\frac{1}{6}x_3x_5^{10}x_6+\frac{5}{6}x_3x_5^9x_6^2-\frac{5}{2}x_3x_5^8x_6^3-\frac{235}{18}x_3x_5^7x_6^4-\frac{65}{9}x_3x_5^6x_6^5+\frac{1297}{6}x_3x_5^5x_6^6-$$
$$\frac{1675}{9}x_3x_5^4x_6^7-\frac{15101}{18}x_3x_5^3x_6^8+\frac{16111}{9}x_3x_5^2x_6^9-\frac{11467}{9}x_3x_5x_6^{10}+\frac{2704}{9}x_3x_6^{11}-\frac{1}{18}x_4x_5^{10}x_6-\frac{4}{9}x_4x_5^9x_6^2+$$
$$\frac{2}{3}x_4x_5^8x_6^3+\frac{43}{18}x_4x_5^7x_6^4+\frac{55}{18}x_4x_5^6x_6^5+\frac{83}{3}x_4x_5^5x_6^6-\frac{509}{2}x_4x_5^4x_6^7+\frac{9991}{18}x_4x_5^3x_6^8-\frac{1400}{3}x_4x_5^2x_6^9+$$
$$\frac{391}{3}x_4x_5x_6^{10}+\frac{1}{9}x_5^9x_6^3+\frac{13}{9}x_5^8x_6^4-\frac{103}{6}x_5^7x_6^5+13x_5^6x_6^6+\frac{2297}{9}x_5^5x_6^7-\frac{8828}{9}x_5^4x_6^8+$$
$$\frac{13819}{9}x_5^3x_6^9-\frac{9934}{9}x_5^2x_6^{10}+\frac{938}{3}x_5x_6^{11} \qquad (h_{25})$$

$$x_3^3x_5^4x_6^6-\frac{766618}{133145}x_3^3x_5^3x_6^7+\frac{308818}{26629}x_3^3x_5^2x_6^8-\frac{1313024}{133145}x_3^3x_5x_6^9+\frac{461072}{133145}x_3^3x_6^{10}+\frac{9}{133145}x_3^2x_4x_5^{10}+$$
$$\frac{362}{133145}x_3^2x_4x_5^9x_6+\frac{76}{4295}x_3^2x_4x_5^8x_6^2+\frac{206}{133145}x_3^2x_4x_5^7x_6^3-\frac{27572}{133145}x_3^2x_4x_5^6x_6^4+\frac{92099}{133145}x_3^2x_4x_5^5x_6^5-$$
$$\frac{417576}{133145}x_3^2x_4x_5^4x_6^6+\frac{1006392}{133145}x_3^2x_4x_5^3x_6^7-\frac{1232824}{133145}x_3^2x_4x_5^2x_6^8+\frac{933676}{133145}x_3^2x_4x_5x_6^9-\frac{442752}{133145}x_3^2x_4x_6^{10}$$
$$-\frac{18}{133145}x_3^2x_5^{10}x_6-\frac{706}{133145}x_3^2x_5^9x_6^2-\frac{776}{26629}x_3^2x_5^8x_6^3+\frac{1430}{26629}x_3^2x_5^7x_6^4+\frac{49521}{133145}x_3^2x_5^6x_6^5+\frac{54460}{26629}x_3^2x_5^5x_6^6-$$
$$\frac{1597184}{133145}x_3^2x_5^4x_6^7+\frac{302308}{26629}x_3^2x_5^3x_6^8+\frac{1653476}{133145}x_3^2x_5^2x_6^9-\frac{2880296}{133145}x_3^2x_5x_6^{10}+\frac{1247424}{133145}x_3^2x_6^{11}+$$
$$\frac{9}{133145}x_3x_4x_5^{11}+\frac{362}{133145}x_3x_4x_5^{10}x_6+\frac{2221}{133145}x_3x_4x_5^9x_6^2-\frac{890}{26629}x_3x_4x_5^8x_6^3-\frac{6923}{26629}x_3x_4x_5^7x_6^4+$$
$$\frac{182959}{133145}x_3x_4x_5^6x_6^5-\frac{708272}{133145}x_3x_4x_5^5x_6^6+\frac{38871}{4295}x_3x_4x_5^4x_6^7+\frac{273510}{26629}x_3x_4x_5^3x_6^8-\frac{6324426}{133145}x_3x_4x_5^2x_6^9+$$
$$\frac{5910408}{133145}x_3x_4x_5x_6^{10}-\frac{1854928}{133145}x_3x_4x_6^{11}-\frac{27}{133145}x_3x_5^{11}x_6-\frac{201}{26629}x_3x_5^{10}x_6^2-\frac{789}{26629}x_3x_5^9x_6^3+\frac{3033}{26629}x_3x_5^8x_6^4+$$
$$\frac{14188}{26629}x_3x_5^7x_6^5+\frac{6083}{133145}x_3x_5^6x_6^6-\frac{1103586}{133145}x_3x_5^5x_6^7+\frac{1065451}{133145}x_3x_5^4x_6^8+\frac{817502}{26629}x_3x_5^3x_6^9-\frac{8989582}{133145}x_3x_5^2x_6^{10}+$$
$$\frac{6448288}{133145}x_3x_5x_6^{11}-\frac{1525552}{133145}x_3x_6^{12}+\frac{9}{133145}x_4x_5^{11}x_6+\frac{362}{133145}x_4x_5^{10}x_6^2+\frac{2212}{133145}x_4x_5^9x_6^3-\frac{3867}{133145}x_4x_5^8x_6^4-$$
$$\frac{13451}{133145}x_4x_5^7x_6^5-\frac{22214}{133145}x_4x_5^6x_6^6-\frac{20541}{26629}x_4x_5^5x_6^7+\frac{1238571}{133145}x_4x_5^4x_6^8-\frac{2791496}{133145}x_4x_5^3x_6^9+\frac{2367906}{133145}x_4x_5^2x_6^{10}-$$
$$\frac{663168}{133145}x_4x_5x_6^{11}-\frac{18}{133145}x_5^{10}x_6^3-\frac{814}{133145}x_5^9x_6^4-\frac{4759}{133145}x_5^8x_6^5+\frac{87504}{133145}x_5^7x_6^6-\frac{106614}{133145}x_5^6x_6^7-\frac{235384}{26629}x_5^5x_6^8+$$
$$\frac{4834238}{133145}x_5^4x_6^9-\frac{7699804}{133145}x_5^3x_6^{10}+\frac{5569804}{133145}x_5^2x_6^{11}-\frac{1582224}{133145}x_5x_6^{12} \qquad (h_{26})$$

$$x_3^3x_5^3x_6^8-\frac{21419704390}{5891784319}x_3^3x_5^2x_6^9+\frac{67629031306}{17675352957}x_3^3x_5x_6^{10}-\frac{26933444528}{17675352957}x_3^3x_6^{11}-\frac{266290}{17675352957}x_3^2x_4x_5^{11}-$$
$$\frac{1398712}{5891784319}x_3^2x_4x_5^{10}x_6-\frac{8844891}{5891784319}x_3^2x_4x_5^9x_6^2-\frac{17193468}{5891784319}x_3^2x_4x_5^8x_6^3+\frac{29823492}{5891784319}x_3^2x_4x_5^7x_6^4+$$
$$\frac{516104788}{17675352957}x_3^2x_4x_5^6x_6^5-\frac{1956984766}{17675352957}x_3^2x_4x_5^5x_6^6+\frac{3346014393}{5891784319}x_3^2x_4x_5^4x_6^7-\frac{28193350168}{17675352957}x_3^2x_4x_5^3x_6^8+$$
$$\frac{17827290872}{5891784319}x_3^2x_4x_5^2x_6^9-\frac{53429894704}{17675352957}x_3^2x_4x_5x_6^{10}+\frac{28813679548}{17675352957}x_3^2x_4x_6^{11}+\frac{532580}{17675352957}x_3^2x_5^{11}x_6+$$
$$\frac{7859692}{17675352957}x_3^2x_5^{10}x_6^2+\frac{13827198}{5891784319}x_3^2x_5^9x_6^3+\frac{43941970}{17675352957}x_3^2x_5^8x_6^4-\frac{222293370}{5891784319}x_3^2x_5^7x_6^5-$$

$$\frac{1100441104}{17675352957}x_3^2x_5^6x_6^6+\frac{2521805215}{17675352957}x_3^2x_5^5x_6^7+\frac{44903378516}{17675352957}x_3^2x_5^4x_6^8-\frac{98116397120}{17675352957}x_3^2x_5^3x_6^9-$$
$$\frac{37194705164}{17675352957}x_3^2x_5^2x_6^{10}+\frac{132441807964}{17675352957}x_3^2x_5x_6^{11}-\frac{68788204376}{17675352957}x_3^2x_6^{12}-\frac{266290}{17675352957}x_3x_4x_5^{12}-$$
$$\frac{1398712}{5891784319}x_3x_4x_5^{11}x_6-\frac{7513441}{5891784319}x_3x_4x_5^{10}x_6^2-\frac{11539304}{17675352957}x_3x_4x_5^9x_6^3+\frac{70168075}{5891784319}x_3x_4x_5^8x_6^4+$$
$$\frac{170902970}{17675352957}x_3x_4x_5^7x_6^5-\frac{3968873371}{17675352957}x_3x_4x_5^6x_6^6+\frac{17853449993}{17675352957}x_3x_4x_5^5x_6^7-\frac{16888092208}{5891784319}x_3x_4x_5^4x_6^8+$$
$$\frac{4304460295}{5891784319}x_3x_4x_5^3x_6^9+\frac{222739621894}{17675352957}x_3x_4x_5^2x_6^{10}-\frac{90703010854}{5891784319}x_3x_4x_5x_6^{11}+\frac{32760384424}{5891784319}x_3x_4x_6^{12}+$$
$$\frac{266290}{5891784319}x_3x_5^{12}x_6+\frac{3397266}{5891784319}x_3x_5^{11}x_6^2+\frac{15277715}{5891784319}x_3x_5^{10}x_6^3-\frac{21395095}{17675352957}x_3x_5^9x_6^4-$$
$$\frac{970450985}{17675352957}x_3x_5^8x_6^5-\frac{87348555}{5891784319}x_3x_5^7x_6^6+\frac{8604140518}{17675352957}x_3x_5^6x_6^7+\frac{21227173829}{17675352957}x_3x_5^5x_6^8-$$
$$\frac{82893014974}{17675352957}x_3x_5^4x_6^9-\frac{80270876345}{17675352957}x_3x_5^3x_6^{10}+\frac{354190470838}{17675352957}x_3x_5^2x_6^{11}-\frac{102197410514}{5891784319}x_3x_5x_6^{12}+$$
$$\frac{26440410016}{5891784319}x_3x_6^{13}-\frac{266290}{17675352957}x_4x_5^{12}x_6-\frac{1398712}{5891784319}x_4x_5^{11}x_6^2-\frac{22274033}{17675352957}x_4x_5^{10}x_6^3-$$
$$\frac{35303618}{17675352957}x_4x_5^9x_6^4+\frac{225171518}{17675352957}x_4x_5^8x_6^5+\frac{331715539}{17675352957}x_4x_5^7x_6^6-\frac{1003512239}{17675352957}x_4x_5^6x_6^7-$$
$$\frac{1997602640}{17675352957}x_4x_5^5x_6^8-\frac{23461414769}{17675352957}x_4x_5^4x_6^9+\frac{95939535619}{17675352957}x_4x_5^3x_6^{10}-\frac{104789169464}{17675352957}x_4x_5^2x_6^{11}+$$
$$\frac{33410175082}{17675352957}x_4x_5x_6^{12}+\frac{532580}{17675352957}x_5^{11}x_6^3+\frac{11055172}{17675352957}x_5^{10}x_6^4-\frac{10686424}{17675352957}x_5^9x_6^5-$$
$$\frac{114502558}{5891784319}x_5^8x_6^6-\frac{1133336681}{17675352957}x_5^7x_6^7+\frac{8863067986}{17675352957}x_5^6x_6^8+\frac{3497464050}{5891784319}x_5^5x_6^9-\frac{137099469272}{17675352957}x_5^4x_6^{10}+$$
$$\frac{103512277402}{5891784319}x_5^3x_6^{11}-\frac{269782907756}{17675352957}x_5^2x_6^{12}+\frac{86448281276}{17675352957}x_5x_6^{13} \tag{h_{27}}$$

$$x_3^3x_5^2x_6^{10}-\frac{21287287080210}{5416918232789}x_3^3x_5x_6^{11}+\frac{22397095401574}{5416918232789}x_3^3x_6^{12}+\frac{5891784319}{21667672931156}x_3^2x_4x_5^{12}+$$
$$\frac{467078765}{125974842623}x_3^2x_4x_5^{11}x_6+\frac{416493985353}{21667672931156}x_3^2x_4x_5^{10}x_6^2+\frac{958556854461}{21667672931156}x_3^2x_4x_5^9x_6^3+$$
$$\frac{89946589623}{773845461827}x_3^2x_4x_5^8x_6^4-\frac{6612836714977}{21667672931156}x_3^2x_4x_5^7x_6^5-\frac{13451599559941}{21667672931156}x_3^2x_4x_5^6x_6^6-$$
$$\frac{21736711235081}{10833836465578}x_3^2x_4x_5^5x_6^7-\frac{32843932627796}{5416918232789}x_3^2x_4x_5^4x_6^8+\frac{112209674624762}{5416918232789}x_3^2x_4x_5^3x_6^9+$$
$$\frac{18092946932965}{5416918232789}x_3^2x_4x_5^2x_6^{10}-\frac{100986848182916}{5416918232789}x_3^2x_4x_5x_6^{11}+\frac{13127250837928}{5416918232789}x_3^2x_4x_6^{12}-$$
$$\frac{5891784319}{10833836465578}x_3^2x_5^{12}x_6-\frac{74445763261}{10833836465578}x_3^2x_5^{11}x_6^2-\frac{300805731859}{10833836465578}x_3^2x_5^{10}x_6^3-$$
$$\frac{78436526015}{1547690923654}x_3^2x_5^9x_6^4+\frac{5421961689925}{21667672931156}x_3^2x_5^8x_6^5+\frac{14035015596157}{21667672931156}x_3^2x_5^7x_6^6+$$
$$\frac{5426522224603}{3095381847308}x_3^2x_5^6x_6^7-\frac{2688980535661}{773845461827}x_3^2x_5^5x_6^8-\frac{102982682846720}{5416918232789}x_3^2x_5^4x_6^9+$$
$$\frac{163885517859078}{5416918232789}x_3^2x_5^3x_6^{10}+\frac{44943281903469}{5416918232789}x_3^2x_5^2x_6^{11}-\frac{25684886021478}{773845461827}x_3^2x_5x_6^{12}+$$
$$\frac{108488186197912}{5416918232789}x_3^2x_6^{13}+\frac{5891784319}{21667672931156}x_3x_4x_5^{13}+\frac{467078765}{125974842623}x_3x_4x_5^{12}x_6+$$
$$\frac{82029305142}{5416918232789}x_3x_4x_5^{11}x_6^2+\frac{260187092195}{21667672931156}x_3x_4x_5^{10}x_6^3+\frac{331052940136}{5416918232789}x_3x_4x_5^9x_6^4-$$
$$\frac{3483765412555}{10833836465578}x_3x_4x_5^8x_6^5-\frac{1458407676261}{5416918232789}x_3x_4x_5^7x_6^6-\frac{43987280728689}{21667672931156}x_3x_4x_5^6x_6^7-$$

$$\frac{30263645638690}{5416918232789}x_3x_4x_5^5x_6^8+\frac{218663520348261}{5416918232789}x_3x_4x_5^4x_6^9-\frac{76268265891892}{5416918232789}x_3x_4x_5^3x_6^{10}-$$
$$\frac{74884224622460}{773845461827}x_3x_4x_5^2x_6^{11}+\frac{602136413868342}{5416918232789}x_3x_4x_5x_6^{12}-\frac{208364916739998}{5416918232789}x_3x_4x_6^{13}-$$
$$\frac{17675352957}{21667672931156}x_3x_5^{13}x_6-\frac{26855226267}{3095381847308}x_3x_5^{12}x_6^2-\frac{3574539492}{125974842623}x_3x_5^{11}x_6^3-$$
$$\frac{326396124587}{21667672931156}x_3x_5^{10}x_6^4+\frac{4439869166885}{10833836465578}x_3x_5^9x_6^5+\frac{4772524801543}{21667672931156}x_3x_5^8x_6^6+$$
$$\frac{5690831382125}{21667672931156}x_3x_5^7x_6^7-\frac{47780503773085}{21667672931156}x_3x_5^6x_6^8-\frac{205342426154217}{10833836465578}x_3x_5^5x_6^9+$$
$$\frac{253296910598890}{5416918232789}x_3x_5^4x_6^{10}+\frac{100022075531127}{5416918232789}x_3x_5^3x_6^{11}-\frac{747392002257722}{5416918232789}x_3x_5^2x_6^{12}+$$
$$\frac{697182100612332}{5416918232789}x_3x_5x_6^{13}-\frac{189072088755474}{5416918232789}x_3x_6^{14}+\frac{5891784319}{21667672931156}x_4x_5^{13}x_6+$$
$$\frac{467078765}{125974842623}x_4x_5^{12}x_6^2+\frac{322225436249}{21667672931156}x_4x_5^{11}x_6^3+\frac{399243449055}{10833836465578}x_4x_5^{10}x_6^4-$$
$$\frac{142612482499}{21667672931156}x_4x_5^9x_6^5-\frac{8124473548763}{21667672931156}x_4x_5^8x_6^6-\frac{432339288513}{5416918232789}x_4x_5^7x_6^7-$$
$$\frac{68169948045657}{21667672931156}x_4x_5^6x_6^8+\frac{61814577820617}{10833836465578}x_4x_5^5x_6^9+\frac{206797018628895}{10833836465578}x_4x_5^4x_6^{10}-$$
$$\frac{49135832074764}{773845461827}x_4x_5^3x_6^{11}+\frac{46263622090152}{773845461827}x_4x_5^2x_6^{12}-\frac{94766982983420}{5416918232789}x_4x_5x_6^{13}-$$
$$\frac{5891784319}{10833836465578}x_5^{12}x_6^3-\frac{109796469175}{10833836465578}x_5^{11}x_6^4+\frac{100382132591}{3095381847308}x_5^{10}x_6^5+$$
$$\frac{2353790698075}{10833836465578}x_5^9x_6^6-\frac{718891622629}{3095381847308}x_5^8x_6^7+\frac{7701874971201}{21667672931156}x_5^7x_6^8-$$
$$\frac{64562755555783}{10833836465578}x_5^6x_6^9+\frac{17300197520264}{5416918232789}x_5^5x_6^{10}+\frac{222704634223819}{5416918232789}x_5^4x_6^{11}-$$
$$\frac{516356292293944}{5416918232789}x_5^3x_6^{12}+\frac{444787984530968}{5416918232789}x_5^2x_6^{13}-\frac{21003236785330}{773845461827}x_5x_6^{14} \qquad (h_{28})$$

$$x_3^3x_5x_6^{12}-\frac{10236206889673892}{3167065052629021}x_3^3x_6^{13}-\frac{16250754698367}{50673040842064336}x_3^2x_4x_5^{13}-\frac{56592352056075}{12668260210516084}x_3^2x_4x_5^{12}x_6-$$
$$\frac{1214880338156689}{50673040842064336}x_3^2x_4x_5^{11}x_6^2-\frac{2996121101956077}{50673040842064336}x_3^2x_4x_5^{10}x_6^3-\frac{976782439961343}{6334130105258042}x_3^2x_4x_5^9x_6^4+$$
$$\frac{16021087156345425}{50673040842064336}x_3^2x_4x_5^8x_6^5+\frac{42771926829399237}{50673040842064336}x_3^2x_4x_5^7x_6^6+\frac{66285663895320089}{25336520421032168}x_3^2x_4x_5^6x_6^7+$$
$$\frac{48878546862263911}{6334130105258042}x_3^2x_4x_5^5x_6^8-\frac{274355362254010407}{12668260210516084}x_3^2x_4x_5^4x_6^9-\frac{164807809294276573}{12668260210516084}x_3^2x_4x_5^3x_6^{10}+$$
$$\frac{77266528304222391}{3167065052629021}x_3^2x_4x_5^2x_6^{11}+\frac{147811100261498}{3167065052629021}x_3^2x_4x_5x_6^{12}+\frac{3425457977880727}{3167065052629021}x_3^2x_4x_6^{13}+$$
$$\frac{16250754698367}{25336520421032168}x_3^2x_5^{13}x_6+\frac{210118653525933}{25336520421032168}x_3^2x_5^{12}x_6^2+\frac{891006401742187}{25336520421032168}x_3^2x_5^{11}x_6^3+$$
$$\frac{1771836954418049}{25336520421032168}x_3^2x_5^{10}x_6^4-\frac{13923377667998589}{50673040842064336}x_3^2x_5^9x_6^5-\frac{42963473298446021}{50673040842064336}x_3^2x_5^8x_6^6-$$
$$\frac{118418251463416773}{50673040842064336}x_3^2x_5^7x_6^7+\frac{43279629239383607}{12668260210516084}x_3^2x_5^6x_6^8+\frac{301456192455026455}{12668260210516084}x_3^2x_5^5x_6^9-$$
$$\frac{165245062961806889}{6334130105258042}x_3^2x_5^4x_6^{10}-\frac{339913793181074957}{12668260210516084}x_3^2x_5^3x_6^{11}+\frac{213995889403597607}{6334130105258042}x_3^2x_5^2x_6^{12}-$$
$$\frac{103386705183877743}{3167065052629021}x_3^2x_5x_6^{13}-\frac{36549530601448454}{3167065052629021}x_3^2x_6^{14}-\frac{16250754698367}{50673040842064336}x_3x_4x_5^{14}-$$
$$\frac{56592352056075}{12668260210516084}x_3x_4x_5^{13}x_6-\frac{60694938605074}{3167065052629021}x_3x_4x_5^{12}x_6^2-\frac{998144882651139}{50673040842064336}x_3x_4x_5^{11}x_6^3-$$

$$\frac{984958956161151}{12668260210516084}x_3x_4x_5^{10}x_6^4+\frac{9052054197319547}{25336520421032168}x_3x_4x_5^9x_6^5+\frac{2807139584154699}{6334130105258042}x_3x_4x_5^8x_6^6+$$
$$\frac{126638543442217817}{50673040842064336}x_3x_4x_5^7x_6^7+\frac{89043745990018979}{12668260210516084}x_3x_4x_5^6x_6^8-\frac{140866212687651025}{3167065052629021}x_3x_4x_5^5x_6^9-$$
$$\frac{2483789606234187}{3167065052629021}x_3x_4x_5^4x_6^{10}+\frac{1518639530287417425}{12668260210516084}x_3x_4x_5^3x_6^{11}-\frac{261772600035128168}{3167065052629021}x_3x_4x_5^2x_6^{12}-$$
$$\frac{36001967141396937}{3167065052629021}x_3x_4x_5x_6^{13}+\frac{63047678363345478}{3167065052629021}x_3x_4x_6^{14}+\frac{48752264095101}{50673040842064336}x_3x_5^{14}x_6+$$
$$\frac{532851432387597}{50673040842064336}x_3x_5^{13}x_6^2+\frac{231384707615859}{6334130105258042}x_3x_5^{12}x_6^3+\frac{1433730631124163}{50673040842064336}x_3x_5^{11}x_6^4-$$
$$\frac{12036132919466475}{25336520421032168}x_3x_5^{10}x_6^5-\frac{20442942845913379}{50673040842064336}x_3x_5^9x_6^6-\frac{22851262521868385}{50673040842064336}x_3x_5^8x_6^7$$
$$+\frac{126283635323639145}{50673040842064336}x_3x_5^7x_6^8+\frac{600312732302041277}{25336520421032168}x_3x_5^6x_6^9-\frac{597371455589205061}{12668260210516084}x_3x_5^5x_6^{10}$$
$$-\frac{551532422692500325}{12668260210516084}x_3x_5^4x_6^{11}+\frac{1912427945114215321}{12668260210516084}x_3x_5^3x_6^{12}-\frac{508487231860963939}{6334130105258042}x_3x_5^2x_6^{13}-$$
$$\frac{75877831923360765}{3167065052629021}x_3x_5x_6^{14}+\frac{55126349217435072}{3167065052629021}x_3x_6^{15}-\frac{16250754698367}{50673040842064336}x_4x_5^{14}x_6-$$
$$\frac{56592352056075}{12668260210516084}x_4x_5^{13}x_6^2-\frac{954868262982817}{50673040842064336}x_4x_5^{12}x_6^3-\frac{1239052358877687}{25336520421032168}x_4x_5^{11}x_6^4-$$
$$\frac{330028285668545}{50673040842064336}x_4x_5^{10}x_6^5+\frac{22405350369039427}{50673040842064336}x_4x_5^9x_6^6+\frac{3031319659456187}{12668260210516084}x_4x_5^8x_6^7+$$
$$\frac{190551939623864197}{50673040842064336}x_4x_5^7x_6^8-\frac{144529744089868399}{25336520421032168}x_4x_5^6x_6^9-\frac{623936020585737295}{25336520421032168}x_4x_5^5x_6^{10}+$$
$$\frac{844934728525573555}{12668260210516084}x_4x_5^4x_6^{11}-\frac{532976275285939373}{12668260210516084}x_4x_5^3x_6^{12}-\frac{22507900435229951}{3167065052629021}x_4x_5^2x_6^{13}+$$
$$\frac{52578318193859501}{6334130105258042}x_4x_5x_6^{14}+\frac{16250754698367}{25336520421032168}x_5^{13}x_6^3+\frac{307623181716135}{25336520421032168}x_5^{12}x_6^4-$$
$$\frac{1758094856695321}{50673040842064336}x_5^{11}x_6^5-\frac{6758703075008731}{25336520421032168}x_5^{10}x_6^6+\frac{10023046298568323}{50673040842064336}x_5^9x_6^7-$$
$$\frac{18320545460988729}{50673040842064336}x_5^8x_6^8+\frac{173035851475310717}{25336520421032168}x_5^7x_6^9-\frac{7012632182135623}{6334130105258042}x_5^6x_6^{10}$$
$$-\frac{619437262342931877}{12668260210516084}x_5^5x_6^{11}+\frac{281893683861330460}{3167065052629021}x_5^4x_6^{12}-\frac{272616852556601949}{6334130105258042}x_5^3x_6^{13}-$$
$$\frac{91764328515154903}{6334130105258042}x_5^2x_6^{14}+\frac{48100407330654959}{3167065052629021}x_5x_6^{15} \qquad (h_{29})$$

$$x_3^3x_6^{14}+\frac{3167065052629021}{18562435800941653920}x_3^2x_4x_5^{14}+\frac{2328158745331339}{1031246433385647440}x_3^2x_4x_5^{13}x_6+$$
$$\frac{160515378503938 03}{1427879676995511840}x_3^2x_4x_5^{12}x_6^2+\frac{91066224776995025}{3712487160188330784}x_3^2x_4x_5^{11}x_6^3+\frac{217555995230307503}{3093739300156942320}x_3^2x_4x_5^{10}x_6^4-$$
$$\frac{1249315513658304301}{6187478600313884640}x_3^2x_4x_5^9x_6^5-\frac{428154758322618 97}{158653297443945760}x_3^2x_4x_5^8x_6^6-\frac{319925068714695911}{257811608346411860}x_3^2x_4x_5^7x_6^7-$$
$$\frac{44660596529502139}{12890580417320593}x_3^2x_4x_5^6x_6^8+\frac{1236142926564708 9295}{928121790047082696}x_3^2x_4x_5^5x_6^9-\frac{155040099686068 6819}{356969919248877960}x_3^2x_4x_5^4x_6^{10}-$$
$$\frac{5250049300727637593}{773434825039235580}x_3^2x_4x_5^3x_6^{11}+\frac{2507777196510667387}{232030447511770674}x_3^2x_4x_5^2x_6^{12}-\frac{12154237650351515101}{1160152237558853370}x_3^2x_4x_5x_6^{13}+$$
$$\frac{980764134156930067}{580076118779426685}x_3^2x_4x_6^{14}-\frac{3167065052629021}{9281217900470826960}x_3^2x_5^{14}x_6-\frac{38739792363335081}{9281217900470826960}x_3^2x_5^{13}x_6^2-$$
$$\frac{147760744323381211}{9281217900470826960}x_3^2x_5^{12}x_6^3-\frac{258086386702279817}{9281217900470826960}x_3^2x_5^{11}x_6^4+\frac{584226650879959543}{3712487160188330784}x_3^2x_5^{10}x_6^5+$$
$$\frac{5979347682997283677}{18562435800941653920}x_3^2x_5^9x_6^6+\frac{3941507501028649165}{3712487160188330784}x_3^2x_5^8x_6^7-\frac{4344344825216034653}{1856243580094165392}x_3^2x_5^7x_6^8-$$

$$\frac{302373074884313}{28909848929949}x_3^2x_5^6x_6^9+\frac{40341305166665164472}{193358706259808895}x_3^2x_5^5x_6^{10}-\frac{25629389019982006301}{4640608950235413480}x_3^2x_5^4x_6^{11}-$$
$$\frac{13733641761421335839}{1160152237558853370}x_3^2x_5^3x_6^{12}+\frac{21684634861638666371}{1160152237558853370}x_3^2x_5^2x_6^{13}-\frac{2461508643111772927}{193358706259808895}x_3^2x_5x_6^{14}+$$
$$\frac{3695875767180565054}{580076118779426685}x_3^2x_6^{15}+\frac{3167065052629021}{18562435800941653920}x_3x_4x_5^{15}+\frac{2328158745331339}{10312464333385647440}x_3x_4x_5^{14}x_6+$$
$$\frac{40291004066421031}{4640608950235413480}x_3x_4x_5^{13}x_6^2+\frac{11010650796845489}{2062492866771294880}x_3x_4x_5^{12}x_6^3+\frac{26055011224994153}{618747860031388464}x_3x_4x_5^{11}x_6^4-$$
$$\frac{1972691698585531819}{9281217900470826960}x_3x_4x_5^{10}x_6^5-\frac{327851285773536007}{4640608950235413480}x_3x_4x_5^9x_6^6-\frac{2755810761633290667}{2062492866771294880}x_3x_4x_5^8x_6^7-$$
$$\frac{9142235321382556507}{3093739300156942320}x_3x_4x_5^7x_6^8+\frac{116907575752387704103}{4640608950235413480}x_3x_4x_5^6x_6^9-\frac{22118211140254714999}{1160152237558853370}x_3x_4x_5^5x_6^{10}-$$
$$\frac{65884526593283583493}{1546869650078471160}x_3x_4x_5^4x_6^{11}+\frac{1015667188024331633}{12542186351987604}x_3x_4x_5^3x_6^{12}-\frac{88305360015687741601}{1160152237558853370}x_3x_4x_5^2x_6^{13}+$$
$$\frac{659318475117270209}{14873746635369915}x_3x_4x_5x_6^{14}-\frac{7584499403796790927}{580076118779426685}x_3x_4x_6^{15}-\frac{3167065052629021}{6187478600313884640}x_3x_5^{15}x_6-$$
$$\frac{10801887419359013}{2062492866771294880}x_3x_5^{14}x_6^2-\frac{16464124178395373}{10312464333385647440}x_3x_5^{13}x_6^3-\frac{113399239345682933}{18562435800941653920}x_3x_5^{12}x_6^4+$$
$$\frac{3112234785300305}{12542186351987604}x_3x_5^{11}x_6^5+\frac{562139805270160441}{18562435800941653920}x_3x_5^{10}x_6^6+\frac{5720882040178369589}{18562435800941653920}x_3x_5^9x_6^7-$$
$$\frac{8514343319346308707}{6187478600313884640}x_3x_5^8x_6^8-\frac{6694430520584321221}{580076118779426685}x_3x_5^7x_6^9+\frac{50596282234337479039}{1546869650078471160}x_3x_5^6x_6^{10}-$$
$$\frac{20534184790067497697}{4640608950235413480}x_3x_5^5x_6^{11}-\frac{332255328026835629053}{4640608950235413480}x_3x_5^4x_6^{12}+\frac{189121752795517171}{1741970326664945}x_3x_5^3x_6^{13}-$$
$$\frac{5830150606704828669}{64452902086602965}x_3x_5^2x_6^{14}+\frac{29189194077630431989}{580076118779426685}x_3x_5x_6^{15}-\frac{1404779673580906775}{116015223755885337}x_3x_6^{16}+$$
$$\frac{3167065052629021}{18562435800941653920}x_4x_5^{15}x_6+\frac{2328158745331339}{10312464333385647440}x_4x_5^{14}x_6^2+\frac{17555216801450567}{2062492866771294880}x_4x_5^{13}x_6^3+$$
$$\frac{16238784595070521}{773434825039235580}x_4x_5^{12}x_6^4-\frac{13974929040198661}{2062492866771294880}x_4x_5^{11}x_6^5-\frac{4047463727007784451}{18562435800941653920}x_4x_5^{10}x_6^6+$$
$$\frac{309225512727091871}{9281217900470826960}x_4x_5^9x_6^7-\frac{39190352381465991163}{18562435800941653920}x_4x_5^8x_6^8+\frac{1566454762788031793}{356969919248877960}x_4x_5^7x_6^9+$$
$$\frac{86410111389217802281}{9281217900470826960}x_4x_5^6x_6^{10}-\frac{824783764036455613}{19831662180493220}x_4x_5^5x_6^{11}+\frac{29542711076070513311}{515623216692823720}x_4x_5^4x_6^{12}-$$
$$\frac{2713941769813693693}{59494986541479660}x_4x_5^3x_6^{13}+\frac{11486430256416784009}{464060895023541348}x_4x_5^2x_6^{14}-\frac{7223443350047991943}{1160152237558853370}x_4x_5x_6^{15}-$$
$$\frac{3167065052629021}{9281217900470826960}x_5^{14}x_6^3-\frac{57742182679109207}{9281217900470826960}x_5^{13}x_6^4+\frac{420916267623841439}{18562435800941653920}x_5^{12}x_6^5+$$
$$\frac{95634716970810443}{773434825039235580}x_5^{11}x_6^6-\frac{3471100030508592841}{18562435800941653920}x_5^{10}x_6^7+\frac{7078886154779230489}{18562435800941653920}x_5^9x_6^8-$$
$$\frac{154336782449784389}{39663324360986440}x_5^8x_6^9+\frac{2562497077492713659}{773434825039235580}x_5^7x_6^{10}+\frac{20820082960372605733}{928121790047082696}x_5^6x_6^{11}-$$
$$\frac{49348574698840979711}{773434825039235580}x_5^5x_6^{12}+\frac{180540473993404367033}{2320304475117706740}x_5^4x_6^{13}-\frac{15206134602566219837}{257811608346411860}x_5^3x_6^{14}+$$
$$\frac{17533832736632991634}{580076118779426685}x_5^2x_6^{15}-\frac{5105943969673183462}{580076118779426685}x_5x_6^{16} \qquad (h_{30})$$

$$x_3^2x_4x_5^{15}+10x_3^2x_4x_5^{14}x_6+25x_3^2x_4x_5^{13}x_6^2-43x_3^2x_4x_5^{12}x_6^3+88x_3^2x_4x_5^{11}x_6^4-2169x_3^2x_4x_5^{10}x_6^5+3147x_3^2x_4x_5^9x_6^6-$$
$$4014x_3^2x_4x_5^8x_6^7-1746x_3^2x_4x_5^7x_6^8+128360x_3^2x_4x_5^6x_6^9-322988x_3^2x_4x_5^5x_6^{10}+169424x_3^2x_4x_5^4x_6^{11}+$$
$$268184x_3^2x_4x_5^3x_6^{12}-409136x_3^2x_4x_5^2x_6^{13}+208096x_3^2x_4x_5x_6^{14}-38368x_3^2x_4x_6^{15}-2x_3^2x_5^{15}x_6-18x_3^2x_5^{14}x_6^2-$$
$$18x_3^2x_5^{13}x_6^3+90x_3^2x_5^{12}x_6^4+1243x_3^2x_5^{11}x_6^5-1503x_3^2x_5^{10}x_6^6+1731x_3^2x_5^9x_6^7-28860x_3^2x_5^8x_6^8-3270x_3^2x_5^7x_6^9+$$
$$300392x_3^2x_5^6x_6^{10}-567068x_3^2x_5^5x_6^{11}+188144x_3^2x_5^4x_6^{12}+491064x_3^2x_5^3x_6^{13}-626704x_3^2x_5^2x_6^{14}+297632x_3^2x_5x_6^{15}-$$

$$
\begin{gathered}
53056x_3^2x_6^{16} + x_3x_4x_5^{16} + 10x_3x_4x_5^{15}x_6 + 10x_3x_4x_5^{14}x_6^2 - 107x_3x_4x_5^{13}x_6^3 + 258x_3x_4x_5^{12}x_6^4 - 1928x_3x_4x_5^{11}x_6^5 + \\
4068x_3x_4x_5^{10}x_6^6 - 8587x_3x_4x_5^9x_6^7 + 5394x_3x_4x_5^8x_6^8 + 188986x_3x_4x_5^7x_6^9 - 630168x_3x_4x_5^6x_6^{10} + \\
372212x_3x_4x_5^5x_6^{11} + 1286072x_3x_4x_5^4x_6^{12} - 2682776x_3x_4x_5^3x_6^{13} + 2186144x_3x_4x_5^2x_6^{14} - 849728x_3x_4x_5x_6^{15} + \\
131008x_3x_4x_6^{16} - 3x_3x_5^{16}x_6 - 21x_3x_5^{15}x_6^2 + 205x_3x_5^{13}x_6^4 + 1356x_3x_5^{12}x_6^5 - 4689x_3x_5^{11}x_6^6 + 4017x_3x_5^{10}x_6^7 - \\
11539x_3x_5^9x_6^8 - 38930x_3x_5^8x_6^9 + 395722x_3x_5^7x_6^{10} - 784424x_3x_5^6x_6^{11} - 59436x_3x_5^5x_6^{12} + 2247800x_3x_5^4x_6^{13} - \\
3471624x_3x_5^3x_6^{14} + 2478992x_3x_5^2x_6^{15} - 883776x_3x_5x_6^{16} + 127360x_3x_6^{17} + x_4x_5^{16}x_6 + 10x_4x_5^{15}x_6^2 + 9x_4x_5^{14}x_6^3 - \\
12x_4x_5^{13}x_6^4 - 327x_4x_5^{12}x_6^5 - 863x_4x_5^{11}x_6^6 + 4364x_4x_5^{10}x_6^7 - 15597x_4x_5^9x_6^8 + 64312x_4x_5^8x_6^9 - 50600x_4x_5^7x_6^{10} - \\
386692x_4x_5^6x_6^{11} + 1267992x_4x_5^5x_6^{12} - 1743648x_4x_5^4x_6^{13} + 1255792x_4x_5^3x_6^{14} - 463728x_4x_5^2x_6^{15} + \\
69296x_4x_5x_6^{16} - 2x_5^{15}x_6^3 - 30x_5^{14}x_6^4 + 247x_5^{13}x_6^5 + 224x_5^{12}x_6^6 - 3235x_5^{11}x_6^7 + 7341x_5^{10}x_6^8 - 31190x_5^9x_6^9 + \\
95250x_5^8x_6^{10} + 28688x_5^7x_6^{11} - 792424x_5^6x_6^{12} + 1951312x_5^5x_6^{13} - 2341352x_5^4x_6^{14} + 1546736x_5^3x_6^{15} - \\
539280x_5^2x_6^{16} + 77728x_5x_6^{17} \qquad (h_{31})
\end{gathered}
$$

$$
\begin{gathered}
x_3^2x_5^{27} + 16x_3^2x_5^{26}x_6 + 80x_3^2x_5^{25}x_6^2 - 50x_3^2x_5^{24}x_6^3 - 729x_3^2x_5^{23}x_6^4 - 6715x_3^2x_5^{22}x_6^5 - 20512x_3^2x_5^{21}x_6^6 + \\
132590x_3^2x_5^{20}x_6^7 + 35646x_3^2x_5^{19}x_6^8 + 566721x_3^2x_5^{18}x_6^9 + 879869x_3^2x_5^{17}x_6^{10} - 30685539x_3^2x_5^{16}x_6^{11} + 79172127x_3^2x_5^{15}x_6^{12} - \\
65665736x_3^2x_5^{14}x_6^{13} - 67357310x_3^2x_5^{13}x_6^{14} + 1842614868x_3^2x_5^{12}x_6^{15} - 9673494964x_3^2x_5^{11}x_6^{16} + \\
21482268432x_3^2x_5^{10}x_6^{17} - 17132229456x_3^2x_5^9x_6^{18} - 18243686496x_3^2x_5^8x_6^{19} + 55907494672x_3^2x_5^7x_6^{20} - \\
49965801568x_3^2x_5^6x_6^{21} + 4739340256x_3^2x_5^5x_6^{22} + 31007276160x_3^2x_5^4x_6^{23} - 32094531264x_3^2x_5^3x_6^{24} + \\
15941305984x_3^2x_5^2x_6^{25} - 4205207552x_3^2x_5x_6^{26} + 476596224x_3^2x_6^{27} + x_3x_4x_5^{27} + 9x_3x_4x_5^{26}x_6 + 31x_3x_4x_5^{25}x_6^2 + \\
19x_3x_4x_5^{24}x_6^3 - 680x_3x_4x_5^{23}x_6^4 - 2758x_3x_4x_5^{22}x_6^5 + 4165x_3x_4x_5^{21}x_6^6 + 3024x_3x_4x_5^{20}x_6^7 + 160602x_3x_4x_5^{19}x_6^8 - \\
30978x_3x_4x_5^{18}x_6^9 - 3924338x_3x_4x_5^{17}x_6^{10} + 15300227x_3x_4x_5^{16}x_6^{11} - 34676283x_3x_4x_5^{15}x_6^{12} + \\
41716746x_3x_4x_5^{14}x_6^{13} + 436558800x_3x_4x_5^{13}x_6^{14} - 3310129620x_3x_4x_5^{12}x_6^{15} + 9623104148x_3x_4x_5^{11}x_6^{16} - \\
10500697952x_3x_4x_5^{10}x_6^{17} - 11780157040x_3x_4x_5^9x_6^{18} + 53402455264x_3x_4x_5^8x_6^{19} - 71418130064x_3x_4x_5^7x_6^{20} + \\
33416517760x_3x_4x_5^6x_6^{21} + 29312620992x_3x_4x_5^5x_6^{22} - 59751057024x_3x_4x_5^4x_6^{23} + 46382501824x_3x_4x_5^3x_6^{24} - \\
20198893440x_3x_4x_5^2x_6^{25} + 4883773440x_3x_4x_5x_6^{26} - 516052992x_3x_4x_6^{27} + x_3x_5^{28} + 17x_3x_5^{27}x_6 + 72x_3x_5^{26}x_6^2 - \\
203x_3x_5^{25}x_6^3 - 853x_3x_5^{24}x_6^4 - 5990x_3x_5^{23}x_6^5 - 13951x_3x_5^{22}x_6^6 + 177689x_3x_5^{21}x_6^7 - 65712x_3x_5^{20}x_6^8 + \\
32750x_3x_5^{19}x_6^9 + 740830x_3x_5^{18}x_6^{10} - 27974878x_3x_5^{17}x_6^{11} + 83464758x_3x_5^{16}x_6^{12} - 8588245x_3x_5^{15}x_6^{13} - \\
299276902x_3x_5^{14}x_6^{14} + 1166771178x_3x_5^{13}x_6^{15} - 4412141896x_3x_5^{12}x_6^{16} + 7904290356x_3x_5^{11}x_6^{17} + \\
5036433672x_3x_5^{10}x_6^{18} - 48362340208x_3x_5^9x_6^{19} + 91063664096x_3x_5^8x_6^{20} - 73533713008x_3x_5^7x_6^{21} - \\
7736436000x_3x_5^6x_6^{22} + 81479792416x_3x_5^5x_6^{23} - 91704338048x_3x_5^4x_6^{24} + 56054717376x_3x_5^3x_6^{25} - \\
20625538176x_3x_5^2x_6^{26} + 4317667328x_3x_5x_6^{27} - 397682688x_3x_6^{28} - 2x_4x_5^{27}x_6 - 28x_4x_5^{26}x_6^2 - 54x_4x_5^{25}x_6^3 + \\
75x_4x_5^{24}x_6^4 + 305x_4x_5^{23}x_6^5 + 11731x_4x_5^{22}x_6^6 - 8276x_4x_5^{21}x_6^7 - 98228x_4x_5^{20}x_6^8 + 313157x_4x_5^{19}x_6^9 - \\
2359178x_4x_5^{18}x_6^{10} + 6327347x_4x_5^{17}x_6^{11} + 10704162x_4x_5^{16}x_6^{12} - 98209674x_4x_5^{15}x_6^{13} + 350718155x_4x_5^{14}x_6^{14} - \\
853446204x_4x_5^{13}x_6^{15} + 518678354x_4x_5^{12}x_6^{16} + 4541027644x_4x_5^{11}x_6^{17} - 17122544200x_4x_5^{10}x_6^{18} + \\
29491318016x_4x_5^9x_6^{19} - 25720329968x_4x_5^8x_6^{20} + 2400237536x_4x_5^7x_6^{21} + 22002949552x_4x_5^6x_6^{22} - \\
28633144960x_4x_5^5x_6^{23} + 19227916704x_4x_5^4x_6^{24} - 7689954240x_4x_5^3x_6^{25} + 1743419392x_4x_5^2x_6^{26} - \\
173575168x_4x_5x_6^{27} + x_5^{28}x_6 + 14x_5^{27}x_6^2 + 40x_5^{26}x_6^3 - 89x_5^{25}x_6^4 - 691x_5^{24}x_6^5 - 4457x_5^{23}x_6^6 - 1402x_5^{22}x_6^7 + \\
86326x_5^{21}x_6^8 - 5266x_5^{20}x_6^9 + 648x_5^{19}x_6^{10} - 662451x_5^{18}x_6^{11} - 12024580x_5^{17}x_6^{12} + 52982417x_5^{16}x_6^{13} - \\
48099513x_5^{15}x_6^{14} - 71614394x_5^{14}x_6^{15} + 559992998x_5^{13}x_6^{16} - 4550062704x_5^{12}x_6^{17} + 20100197840x_5^{11}x_6^{18} - \\
48990852808x_5^{10}x_6^{19} + 68183588144x_5^9x_6^{20} - 43479100512x_5^8x_6^{21} - 21986722480x_5^7x_6^{22} + \\
77697750976x_5^6x_6^{23} - 83540176032x_5^5x_6^{24} + 52161819072x_5^4x_6^{25} - \\
20117364224x_5^3x_6^{26} + 4487781376x_5^2x_6^{27} - 447176704x_5x_6^{28} \qquad (h_{45})
\end{gathered}
$$